RELATIVE CUBULATION OF SMALL CANCELLATION FREE PRODUCTS

EDUARD EINSTEIN AND THOMAS NG

ABSTRACT. We expand the class of groups with relatively geometric actions on CAT(0) cube complexes by proving that it is closed under $C'(\frac{1}{6})$–small cancellation free products. We build upon a result of Martin and Steenbock who prove an analogous result in the more specialized setting of groups acting properly and cocompactly on CAT(0) cube complexes. Our methods make use of the same blown-up complex of groups to construct a candidate collection of walls. However, rather than arguing geometrically, we show relative cubulation by appealing to a boundary separation criterion and proving that wall stabilizers form a sufficiently rich family of full relatively quasi-convex codimension-one subgroups. The additional flexibility of relatively geometric actions has surprising new applications. In particular, we prove that $C'(\frac{1}{6})$–small cancellation free products of residually finite groups are residually finite.

1. INTRODUCTION

In is unknown whether (relatively) hyperbolic groups are residually finite (even when the peripheral subgroups are residually finite) [H⁺95, Problem 22], [BMS02, Q 5.H1], [Bes04, Q 1.15], and [AS23, Open Problem 1]. On the other hand, work of Agol shows that hyperbolic groups acting properly and cocompactly on CAT(0) cube complexes are residually finite [Ago13, Theorem 1.1]. This article is concerned with the more general context of certain improper actions by relatively hyperbolic groups on CAT(0) cube complexes called **relatively geometric actions** developed by Einstein and Groves [EG20]. The additional cubical geometry makes studying residual properties of relatively cubulated groups more tractable than for arbitrary relatively hyperbolic groups [EG22, GM22]. In particular, we use these ideas to prove the following:

Theorem 1.1. *Let* $A_1, \ldots, A_n$ *be finitely generated residually finite groups with* $n \geq 2$. *Any* $C'(\frac{1}{6})$– *small cancellation free product of* $A_1, \ldots, A_n$ *by a finite set of relators is residually finite.*

We note that Theorem 1.1 has no requirement that the groups A_i are cubulated. Theorem 1.1 witnesses the versatility of relative cubulations.

Recently, Martin and Steenbock showed that the class of groups acting properly and cocompactly on CAT(0) cube complexes is closed under taking $C'(\frac{1}{6})$–small cancellation free products (see Definition 2.9 for a formal definition) [MS17]. Jankiewicz and Wise gave an alternate proof of the same result in the slightly more restricted setting of $C'(\frac{1}{20})$–small cancellation free products who apply their results to produce examples of cocompactly cubulated groups that do not virtually split as a graph of groups.

In [EG20], the first author and Groves devised a relatively geometric boundary cubulation criterion based on a relatively hyperbolic (proper and cocompact) cubulation criterion for relatively hyperbolic groups in [BW12]. The criterion in [EG20] requires peripherals to be one-ended. Here, we construct relatively geometric actions using [EMN24, Theorem 1.3], a stronger version of [EG20, Theorem 2.6] proved by the authors and Suraj Krishna MS that accommodates arbitrary finitely generated peripherals.

Our main result is the following generalization of [MS17] to relatively geometric actions.

Theorem 1.2. *Let $(A_i, \mathcal{P}_i)$ be a collection of $n \geq 2$ finitely generated groups that each act relatively geometrically on $\mathrm{CAT}(0)$ cube complexes, and let $\mathcal{P} = P_1 \cup \cdots \cup P_n$. If G is a $C'(\frac{1}{6})$–free product of $A_1, \ldots, A_n$ by a finite set of relators, then $(G, \mathcal{P})$ acts relatively geometrically on a $\mathrm{CAT}(0)$ cube complex.*

Theorem 1.2 provides a wealth of new examples of groups with relatively geometric actions on CAT(0) cube complexes.

Small cancellation theory has roots in Dehn's solution to the word problem for closed hyperbolic surface groups [Deh87]. Initial work of Tartakovskiĭ [Tar49], Greendlinger [Gre60b, Gre60a], Lyndon and Schupp [LS01], among others formalized the notion of $C'(\frac{1}{6})$ groups, which provides a rich source of word hyperbolic groups. A construction of Rips produces important examples of (residually finite) $C'(\frac{1}{6})$ groups with exotic quotients [Rip82, Wis03]. In particular, small cancellation techniques used in the Rips construction gave rise to examples of torsion-free groups without the unique product property [GMS15, RS87, Ste15]. Arzhantseva and Steenbock extended these ideas to the setting of $C'(\frac{1}{6})$–small cancellation free products [AS23].

$C'(\frac{1}{6})$–small cancellation free products generalize classical $C'(\frac{1}{6})$ groups by allowing factor groups other than $\mathbb{Z}$. Basic examples include one-relator quotients of the form $A * B / \langle\!\langle r \rangle\!\rangle$ where A, B are groups and r is a word in $A * B$ satisfying a technical overlap condition discussed in detail in Section 2. Small cancellation free products have several additional applications. For example they can be used to solve embedding problems for infinite groups [MS71, Sch76] and to construct examples of non-degenerate hyperbolically embedded subgroups of non-relatively hyperbolic groups [GS18, Theorem 6.5]. Most importantly for us, the factors of $C'(\frac{1}{6})$–small cancellation free products embed and the resulting group is hyperbolic relative to the factors [Pan99] (see also [Ste15, Theorem 1] and Proposition 3.6 which follows from [LS01]).

Residual properties of hyperbolic groups acting on CAT(0) cube complexes, especially separability of quasiconvex subgroups, played a central role in Agol's resolution of the Virtual Haken Conjecture [Ago13] and the Virtual Fibering Conjecture [Ago08] for closed hyperbolic 3-manifolds. Using cubical techniques, Wise proved virtual fibering for finite-volume hyperbolic 3-manifolds [Wis21]. There have been many other efforts to extend cubical machinery to relatively hyperbolic groups, e.g. [Ein19, HW14, HW15, Rey23, SW15], but most of these approaches have focused on proper and cocompact or cosparse actions of groups on CAT(0) cube complexes.

Groups that act on CAT(0) cube complexes may fail to be residually finite and can even have no non-trivial quotients. Indeed, there are examples of non-residually finite groups and infinite simple groups that act properly and cocompactly on products of simplicial trees [BM14, Wis07] (see also [Cap19] and references therein). On the other hand, groups acting relatively geometrically on CAT(0) cube complexes have favorable residual properties when the peripheral subgroups are residually finite, see Theorem 2.8 from [EG22] below. Theorem 1.2 and Theorem 2.8 together imply:

Corollary 1.3. *Let G be a small cancellation free product as in Theorem 1.2. If each element of $\mathcal{P}_i$ is residually finite for all i, then G is residually finite and all full relatively quasiconvex subgroups are separable.*

Theorem 1.1 is much stronger than similar residual finiteness results that follow from [MS17, JW22] because the relatively geometric machinery requires fewer restrictions on the factor groups. Theorem 1.1 is a special case of Corollary 1.3, see Section 2.2 for a proof.

We hope this article also serves as an advertisement for constructing relative cubulations using the boundary cubulation criteria from [EG20, EMN24]. While Martin and Steenbock must explicitly construct a single wall space to cubulate in [MS17], these relatively geometric criteria allow us to build a sufficient collection of codimension-1 subgroups for relative cubulation from simultaneously considering two different actions.

1.1. **Structure of this paper:** The proof of Theorem 1.2 for general $C'(\frac{1}{6})$–small cancellation free products is not substantially different from the proof for examples of the form $G = A * B / \langle\!\langle w \rangle\!\rangle$. For ease of notation we restrict ourselves to this special case and remark where additional work is necessary if it is not obvious.

In Section 2, we record some basic facts about relatively geometric actions and small cancellation free products. In particular, we describe a boundary criterion Theorem 2.7 for relative cubulation of Krishna MS and the authors, the relationship between relative geometric actions and residual finiteness Theorem 2.8, and the proofs of both Corollary 1.3 and Theorem 1.1 assuming the statement of Theorem 1.2.

Our proof of Theorem 1.2 relies on a blow-up construction for non-positively curved complexes of groups originating in work of Martin [Mar14] that Martin and Steenbock also employed in the setting of $C'(\frac{1}{6})$–small cancellation free products in [MS17]. We review details of this construction in Section 3. Specifically, we recall a developable complex of groups structure for G in Section 3.1, relative hyperbolicity and acylindricity of the action in Sections 3.2 and 3.3, and the blown-up complex $\mathcal{E}G$ in Section 3.4.

In Section 4, we introduce **hyperstructures** (see Definition 4.2), and study properties of the hyperstructures on $\mathcal{E}G_{bal}$ and X_{bal}, which are subdivisions of $\mathcal{E}G$ and the development of the complex of groups from Section 3 respectively. In [MS17], the hyperstructures of $\mathcal{E}G_{bal}$ and lifts

of the hyperstructures of X_{bal} to $\mathcal{E}G_{bal}$ were used (with different names) to construct a wallspace on $\mathcal{E}G_{bal}$ that yields a geometric action on a CAT(0) cube complex when the vertex groups are properly and cocompactly cubulated.

Our approach diverges from [MS17] at this point. We develop a general framework for relatively cubulating a relatively hyperbolic group acting on a polygonal complex where each of the vertex stabilizers admits a relatively geometric cubulation. We anticipate this framework will also be useful in forthcoming work for relatively cubulating random quotients of free products. Since our blowup $\mathcal{E}G_{bal}$ is locally infinite, new techniques are needed to turn the hyperstructures into appropriate codimension–1 subgroups.

Our proof relies heavily on techniques for studying **generalized fine graphs** (see Definition 5.4) developed by Groves and the authors in [EGN24], which we review in Section 5. These tools allow us to study Bowditch boundaries using the geometry of the complexes described in Section 3. We describe the relationship between the Bowditch boundary of G with respect to the coarser peripheral structure consisting of the entire factor groups and the Bowditch Boundary of G with respect to the peripheral subgroups of the factor groups using results from [Yan14].

In Section 6, we show that if a relatively hyperbolic pair acts on a complex like X_{bal} so that maximal parabolics stabilize exactly one vertex and edge stabilizers are finite, then any pair of points in the Bowditch boundary can be separated by a finite index subgroup of a hyperstructure stabilizer. We also prove that stabilizers of hyperstructures are full and relatively quasiconvex subgroups. Section 7 is devoted to showing that X_{bal} satisfies the hypotheses of the machinery developed in Section 6.

In Section 8, we describe relatively geometric blowups of polygonal complexes and prove that they admit relatively geometric cubulations. We show that any two boundary points not in the boundary of a vertex stabilizer of the polygonal complex can be separated using results from Section 6. Then we separate points in the boundary of a single vertex stabilizer by extending a hyperplane from the relatively geometric cubulation of the vertex stabilizer to a full relatively quasiconvex codimension–1 subgroup. At this point we will have all the hyppotheses needed to apply Theorem 2.7. Finally, we conclude the paper with the proof of Theorem 1.2: we show that $\mathcal{E}G_{bal}$ is relatively geometric blowup and apply the main result of Section 8.

2. Background

Our main objects of study are relatively hyperbolic groups. Relatively hyperbolic groups generalize the class of word hyperbolic groups in the sense of Gromov [Gro87] to also include fundamental groups of cusped hyperbolic manifolds with finite volume among many other important examples. Heuristically, relatively hyperbolic groups have Cayley graphs that look δ–hyperbolic away from translates of certain subgroups. One way to formalize this is due to Bowditch and uses **fine hyperbolic graphs**.

2.1. **Fine hyperbolic graphs and relative hyperbolicity.** A **circuit** in a graph Γ is an embedded loop. Recall that a graph Γ is **fine** if for every edge e and $n \in \mathbb{N}$, there exist finitely many circuits that pass through e and have length at most n. For example, the coned-off Cayley graph of a relatively hyperbolic group pair is a fine hyperbolic graph. There are many equivalent definitions of relative hyperbolicity, see for example [Hru10]. We will mainly use the following definition for relative hyperbolicity in terms of fine hyperbolic graphs:

Definition 2.1 ([Bow12, Definition 2], written as stated in [Hru10, Definition 3.4 (RH-4)]). *Suppose G acts on a δ-hyperbolic graph Γ with finite edge stabilizers and finitely many G-orbits of edges. If K is fine, and $\mathcal{P}$ is a set of representatives of the conjugacy classes of infinite vertex stabilizers, then $(G, \mathcal{P})$ is a **relatively hyperbolic (group) pair**. Conjugates of subgroups in $\mathcal{P}$ are called **peripheral subgroups** or **maximal parabolic subgroups**.*

In Section 3, we will also briefly use an equivalent definition of relative hyperbolicity given in terms of linear relative isoperimetric function to prove Proposition 3.6.

We will need to construct **full relatively quasiconvex** subgroups of the pair $(G, \mathcal{P})$ to construct relative cubulations. We use the following criterion to show that a subgroup is relatively quasiconvex:

Theorem 2.2 ([MPW11, Theorem 1.7]). *Let $(G, \mathcal{P})$ be a relatively hyperbolic pair acting cocompactly on a fine hyperbolic graph so that every edge stabilizer is finite. A subgroup $H \leq G$ is relatively quasi-convex in $(G, \mathcal{P})$ if and only if H stabilizes a quasi-convex H–cocompact connected subgraph of Γ.*

We will not need to engage further with the technical details of being relatively quasiconvex. Therefore, we omit a formal definition and refer the interested reader to [Hru10, Section 6], which gives several equivalent definitions of relative quasiconvexity.

Fullness is required for our cubulation criterion:

Notation 2.3. *Let G be a group, $H \leq G$ and $g \in G$. Then $H^g = gPg^{-1}$.*

Definition 2.4. *Let $(K, \mathcal{D})$ be a relatively hyperbolic group pair. A subgroup $H \leq K$ is **full** if for all $D \in \mathcal{D}$ and $k \in K$, $|H \cap D^k| = \infty$ implies that $H \cap D^k$ is finite index in D^k.*

2.2. **Relatively geometric actions on** CAT(0) **cube complexes.**   The CAT(0) condition is a metric notion of non-positive curvature. CAT(0) cube complexes comprise an important family of examples where the geometry of the space is naturally encoded by the combinatorics of codimension-one subsets called hyperplanes. For much more on CAT(0) cube complexes see for example [Sag14, Wis12]. Group acting properly and cocompactly on CAT(0) cube complexes, sometimes referred to as *cubulated groups*, have been well-studied. The first author and Groves introduced the following in [EG20].

Definition 2.5. *Let* $(G, \mathcal{P})$ *be a relatively hyperbolic pair where* G *acts by isometries on a CAT(0) cube complex* $\widetilde{X}$. *The action of* $(G, \mathcal{P})$ *is* **relatively geometric (with respect to** $\mathcal{P}$**)** *if:*

(1) *the action of* G *on* $\widetilde{X}$ *is cocompact,*

(2) *every peripheral subgroup* $P \in \mathcal{P}$ *acts elliptically, and*

(3) *cell stabilizers are either finite or conjugate to a finite index subgroup of some* $P \in \mathcal{P}$.

Groups that admit a relatively geometric action with respect to some collection of peripheral subgroups are called **relatively cubulated**.

Given a subgroup $H \leq G$, we write ΛH to denote its limit set. A relatively hyperbolic group may admit a relatively geometric action with respect to only certain peripheral structures. Indeed the action of any group G on a point is a relatively geometric action for $(G, \{G\})$.

Definition 2.6. *Let* $(K, \mathcal{D})$ *be a relatively hyperbolic pair. If there is a collection of subgroups* $\mathcal{D}_0$ *such that each* $D_0 \in \mathcal{D}_0$ *is conjugate into some* $D \in \mathcal{D}$ *and* $(G, \mathcal{D}_0)$ *is relatively hyperbolic, then we say that* $(G, \mathcal{D}_0)$ *is a* **refined peripheral structure** *for* $(G, \mathcal{D})$.

We use the following relatively geometric cubulation criterion to produce a relatively geometric action:

Theorem 2.7. ([EMN24, *Theorem 1.3*], *see also* [EG20, *Theorem 2.6*]) *Let* $(G, \mathcal{P})$ *be a finitely generated relatively hyperbolic and suppose that for every pair of distinct points* p *and* q *in the Bowditch Boundary* $\partial_{\mathcal{P}} G$ *there is a full relatively quasiconvex codimension* 1 *subgroup* H *of* G *so that* p *and* q *lie in* H–*distinct components of* $\partial G \smallsetminus \Lambda H$. *Then there exist finitely many full relatively quasiconvex codimension–1 subgroups of* G *and a refined peripheral structure* $(G, \mathcal{P}')$ *for* $(G, \mathcal{P})$ *so that the action of* $(G, \mathcal{P}')$ *on the dual cube complex is relatively geometric.*

Moreover, if each $P \in \mathcal{P}$ *acts elliptically on the dual cube complex then no refinement is needed, i.e., we may take* $\mathcal{P}' = \mathcal{P}$.

To prove Theorem 1.2, we will provide a sufficient collection of full relatively quasiconvex codimension–1 subgroups of a small-cancellation free product that satisfy the hypotheses of Theorem 2.7. Relatively geometric actions have favorable residual properties when the peripherals are residually finite:

Theorem 2.8 ([EG22, Corollary 1.7], see also [GM22, Theorem 4.7])**.** *Let* $(G, \mathcal{P})$ *act relatively geometrically on a CAT(0) cube complex* $\widetilde{X}$. *If every* $P \in \mathcal{P}$ *is residually finite, then:*

(1) G *is residually finite, and*

(2) *every full relatively quasiconvex subgroup of* G *is separable.*

The statement of Theorem 1.2 implicitly relies on the fact that the resulting small cancellation free product is hyperbolic relative to the union of the peripheral subgroups of the factor groups, see Proposition 3.6 and Proposition 3.8. With Theorem 2.8, Corollary 1.3 follows immediately from Theorem 1.2. Assuming Theorem 1.2, we can then prove Theorem 1.1:

Proof of Theorem 1.1 assuming Theorem 1.2. Observe that $(A_i, \{A_i\})$ is a relatively hyperbolic pair, and the trivial action of $(A_i, \{A_i\})$ on a point is a relatively geometric action. Therefore Theorem 1.2 implies that if G is any $C'(\frac{1}{6})$-small cancellation free product of $A_1, A_2, \ldots, A_n$, then G acts relatively geometrically on a CAT(0) cube complex with respect to the peripheral structure $\{A_1, \ldots, A_n\}$. Therefore Corollary 1.3 implies that G is residually finite. $\square$

2.3. **Small cancellation free products.** Let $A_1, \ldots, A_n$ be groups. Let $F = A_1 * \cdots * A_n$ be the free product. We will refer to the A_i as **vertex groups** or **factor groups**. Recall that every non-trival element $w \in F$ can be written uniquely in a normal form $w = w_1 w_2 w_3 \ldots w_\ell$ where each w_i lies in a vertex group and w_i, w_{i+1} are in distinct vertex groups. Then there is a natural length $|w|$ where $|w| = \ell$. The word w is **cyclically reduced** if whenever $|w| > 1$, $w_1 \neq w_\ell^{-1}$.

A subset $\mathcal{R} \subseteq F$ is **symmetrized** if every $r \in \mathcal{R}$ is cyclically reduced and every cyclically reduced conjugate of $r^{\pm 1}$ lies in $\mathcal{R}$.

Let $G = F / \langle\!\langle \mathcal{R} \rangle\!\rangle$. Adding all cyclically reduced conjugates of elements of $\mathcal{R}$ and their inverses does not change the quotient group G, so we can assume $\mathcal{R}$ is symmetrized whenever it is necessary.

A reduced word $p \in F$ is a **piece** with respect to a symmetrized set $\mathcal{R}$ if there exist $r_1, r_2 \in \mathcal{R}$ (not necessarily distinct) so that

(1) there exist $u, v \in F$ so that $r_1 = pu$ and $r_2 = pv$,

(2) u and v have no common prefix.

If $\mathcal{R}$ is not symmetrized, we say that p is a piece with respect to $\mathcal{R}$ if it is a piece with respect to the smallest symmetrized $\mathcal{R}_0$ containing $\mathcal{R}$.

Definition 2.9. *Let* $A_1, \ldots, A_n$ *be groups and let* $\mathcal{R}$ *be a finite subset of* $F = A_1 * \cdots * A_n$. *The quotient:*

$$F / \langle\!\langle \mathcal{R} \rangle\!\rangle$$

is a $C'(\frac{1}{6})$*-free product of the* A_i *with respect to a finite set of relators* $\mathcal{R}$ *if for any piece* p *and every* $r \in \mathcal{R}$ *such that* $r = pu$ *for some* $u \in F$:

$$|p| < \tfrac{1}{6}|r|.$$

To avoid pathologies, we will also insist that $|r| \geq 6$.

We now shift our attention to $C'(\frac{1}{6})$–small cancellation free products where the vertex groups are each relatively cubulated. As mentioned in the introduction, proofs for general small cancellation free products can often be deduced from the case of two vertex groups and a single relation. To this end, we use the following notation.

Notation 2.10. *Let A and B be relatively hyperbolic groups with peripherals $\mathcal{P}_A$ and $\mathcal{P}_B$ respectively.*

Let EA and EB be CAT(0) cube complexes on which the groups A and B admit relatively geometric actions with respect to $\mathcal{P}_A$ and $\mathcal{P}_B$ respectively.

From now on, G is a $C'(\frac{1}{6})$ free product of groups A, B with respect to a single relation R.

We will review a construction from [MS17] for a complex of groups whose fundamental group is G in Section 3. The main difference between the case with 2 vertex groups and the case with n vertex groups is the construction of the appropriate complex of groups, which we will briefly address in the next section.

3. A complex of groups for G and the construction of its development

We will make use of a developable complex of groups structure for small cancellation free products due to Martin and Steenbock [MS17]. The purpose of describing this construction is to build an action of G on a 2-dimensional polygonal complex X that captures the relative hyperbolicity of G with respect to the peripheral structure $\mathcal{Q}$ consisting of factor groups. The complex X is also the starting point for building a blown-up space that captures the finer relatively hyperbolic structure $(G, \mathcal{P})$ coming from peripherals within the factor groups.

Throughout this section, we illustrate the construction when G is a small cancellation free product of two groups with a single relator for clarity and ease of notation. Specifically, we assume that $G = A * B / \langle\!\langle w^d \rangle\!\rangle$ where w is a cyclically reduced word of length $2N$ in $A * B$ that is not a proper power and $d \geq 0$.

3.1. **The developed complex X, via complex of groups structure for G.** Fundamental groups of developable complexes of groups naturally act on connected and simply connected cell complexes [BH99, Theorem III.$\mathcal{C}$.3.13]. As with graphs of groups, developable complexes of groups structure can be understood as decompositions of a group into subgroups that determine (up to conjugacy) stabilizers of cells in the universal cover of the complex.

Let L be a connected complex consisting of two vertices labeled u_A and u_B with a single edge joining them, and let L' be its barycentric subdivision.

Definition 3.1. *Let $R_{0,simpl}$ be a complex constructed as follows:*

   (1) *Start with a $2N$-gon, label vertices consecutively (with respect to some orientation on the boundary with labels taken modulo $2N$) $v_0, v_1, \ldots, v_{2N-1}$.*

(2) *Add an **apex vertex** to the center of the $2N$-gon and join the apex vertex to each v_i by adding an edge e_i to obtain a cone over a loop on $2N$ edges. Let σ_i denote the triangle that has both e_{i-1}, e_i as edges.*

Construct a new complex K_{simpl} by gluing the vertices of $R_{0,\mathrm{simpl}}$ with even subscripts to u_A, those vertices with odd subscripts to u_B and gluing the edge between v_i and v_{i+1} to the edge joining u_A to u_B as shown in Figure 1. We emphasize that K_{simpl} is compact because it is a finite cell complex.

Definition 3.2. *The **scwol** over K_{simpl} is a small category $\mathcal{K}$ whose objects are the cells of K_{simpl}. There is an arrow between objects from σ_2 to σ_1 whenever σ_1 is a face of σ_2 (reverse inclusion).*

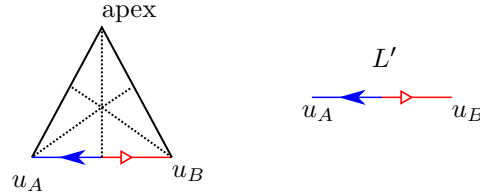See [BH99, Section III.$\mathcal{C}$.1] for a general treatment of scwols.

FIGURE 1. The building blocks of $G(\mathcal{K})$. The underlying complex of $G(\mathcal{K})$ is built from the disjoint union of the two edge graph L' on the right and a polygon consisting of $2N$ copies of the triangle on the left glued together so that the apex is at the center of the polygon. The polygon and L' are glued together as shown.

Definition 3.3. *A **complex of groups** $G(\mathcal{K})$ over a scwol $\mathcal{K}$ is given by the following data:*
  (1) *for each object σ of $\mathcal{K}$, a group G_σ called the **local group**,*
  (2) *for each arrow f, an injective homomorphism $\psi_f : G_{i(f)} \to G_{t(f)}$,*
  (3) *if $t(f_2) = i(f_1)$, there is an element $g_{f_1, f_2} \in G_{t(f_1)}$ so that:*

$$Ad_{g_{f_1,f_2}} \psi_{f_1 f_2} = \psi_{f_1} \psi_{f_2}$$

  *where Ad_g is conjugation by g. The element g_{f_1, f_2} is called a **twisting element**.*
*Additionally, the data must satisfy the **cocycle condition**: for any triple f_1, f_2, f_3 such that $f_1 f_2 f_3$ is a valid composition of arrows in $\mathcal{K}$:*

$$\psi_{f_1}(g_{f_2, f_3}) g_{f_1, f_2 f_3} = g_{f_1, f_2} g_{f_1 f_2, f_3}$$

For much more detail about complexes of groups, see [BH99, Chapter III.$\mathcal{C}$].

Let $\mathcal{K}$ be the scwol over K_{simpl}. Martin and Steenbock construct a complex of groups $G(\mathcal{K})$ over $\mathcal{K}$ where the local groups are trivial except for:
  • The local group at u_A is A,

- the local group at u_B is B,
- and the local group at the apex vertex is $\mathbb{Z}/d\mathbb{Z}$ (recall d is defined so that the relator is w^d).

All local maps are trivial. We define the twisting elements according to a more complicated scheme: first, Martin and Steenbock assign a group element h_f to each arrow f of the scwol $\mathcal{K}$. To write down these elements, we use some additional notation: if the small cancellation relator we are using is w^d where w is written as:

$$w = a_0 b_0 a_1 b_1 \ldots, a_n b_n \qquad a_i \in A, \, b_i \in B, \tag{1}$$

then let w_i be the first i letters of w as written in (1). If σ, σ' are distinct simplices of $G(\mathcal{K})$ (which correspond to vertices of $\mathcal{K}$), and $\sigma \subseteq \sigma'$, then (σ, σ') will be used to denote the arrow in $\mathcal{K}$ from σ' to σ. Also recall the notations v_i, e_i, σ_i from Definition 3.1. We assign the h_f as follows:

- for each $1 \leq i \leq 2N$, $h_{(v_{i-1}, e_{i-1})} = w_{i-1}^{-1}$,
- for each $1 \leq i \leq 2N$, $h_{(v_{i-1}, \sigma_i)} = w_{i-1}^{-1}$,
- for each $1 \leq i \leq 2N$, $h_{(L, \sigma_i)} = w_i^{-1}$,
- for each $1 \leq i \leq 2N$, $h_{(v_i, \sigma_i)} = w_i^{-1}$,
- $h_{e_{2N}, \sigma_{2N}} = w_{2N}^{-1}$,
- and h_f is trivial for all other arrows f.

For any e, f a pair of composable arrows in $\mathcal{K}$, the twisting element $g_{f,e}$ is defined as $g_{f,e} = h_f h_e h_{fe}^{-1}$.

Much like a graph of groups, the complex of groups structure of $G(\mathcal{K})$ carries both algebraic and geometric information about the group G.

Theorem 3.4 ([MS17, Propositions 2.5, 2.8, 2.10]). *There exists a simply connected $C'(\frac{1}{6})$ polygonal complex X with an action of G so that the quotient of the action $_G\backslash X$ is isomorphic as a complex of groups to $G(\mathcal{K})$. Also, the fundamental group (in the category of complexes of groups) of $G(\mathcal{K})$ is isomorphic to G.*

In other words, $G(\mathcal{K})$ is a developable complex of groups with X as its universal cover (as a complex of groups). Since $G(\mathcal{K})$ is a finite complex of groups, the action of G on X is cocompact. We will refer to X as the **developed complex** associated to the complex of groups structure $G(\mathcal{K})$. Our main concern is that X is polygonal and G admits a cocompact action on X, so we will not discuss the fundamental group of a complex of groups further. The interested reader should refer to [BH99, Section III.$\mathcal{C}$.3].

Remark 3.5. *The construction described in this subsection directly generalizes to all $C'(\frac{1}{6})$–small cancellation free products with $n \geq 2$ factors and finitely many relations with the following adjustments. The complex L' will be a star where the local groups of valence 1 vertices are each of the factor groups and all other local groups are trivial. Distinct relations will give distinct labelled polygons $R_{0,simpl}$ whose boundaries are glued to L' to build a complex K_{simpl} that gives rise to a complex of groups over the scwol over K_{simpl} as above.*

3.2. **The relatively hyperbolic structure for G with respect to the vertex groups.** A priori quotients of free products need not inherit relative hyperbolicity from the vertex groups. For example, the direct product of two finitely generated infinite groups $A \times B$ arises as a quotient of the free product $A * B$ by finitely many words. It is thus, essential to our arguments that we restrict our attention to quotients satisfying the $C'(\frac{1}{6})$ small cancellation in Section 2.3.

An important aspect of working with the complex X is that its large-scale geometry is closely related to the relative hyperbolicity of G. The following result is originally due to Pankrat'ev [Pan99] and also proved by Steenbock [Ste15, Theorem 1.1]:

Proposition 3.6. *Let $\mathcal{Q} = \{A, B\}$. Then $(G, \mathcal{Q})$ is a relatively hyperbolic pair.*

Proof. Consider a finite relative presentation $\langle A, B \,|\, R \rangle$ for G relative to $\mathcal{Q}$ (see [Hru10, Section 3.5] for details about finite relative presentations). Let w be a reduced word over the alphabet $A \cup B$ so that $w =_G 1_G$. Since R satisfies the $C'(\frac{1}{6})$ condition, [LS01, Chapter V, Theorem 9.3] implies that there exists a reduced factorization $w = usv$ and a reduced word r over the alphabet $A \cup B$ that is a cyclic conjugate of some $r' \in R$, so that $r = st$ and $|s| > \frac{1}{2}|r|$. By applying Dehn's algorithm (see e.g. [BH99, III.Γ.2.4]), we conclude that:

$$w = r_1' r_2' \dots r_k'$$

where each r_i' is a cyclic conjugate of some $r_i \in R$ and $k \leq |w|$. Hence there exists a linear relative isoperimetric function for the finite relative presentation $\langle A, B \,|\, R \rangle$, so $(G, \mathcal{Q})$ satisfies [Hru10, Definition 3.7 (RH-6)], one of the equivalent definitions given in [Hru10] for a relatively hyperbolic pair. $\qquad\square$

A result of Charney and Crisp [CC07, Theorem 5.1] asserts that any space admitting a cocompact action by a relatively hyperbolic group, in which the peripherals are maximal among subgroups that act with nontrivial fixed point, is equivariantly quasi-isometric to any coned–off Cayley graph. Recall that a **coned-off Cayley graph** for $(G, \mathcal{P})$ is obtained from any finite generated set S of G by attaching a cone over distinct cosets of each $P \in \mathcal{P}$ in $\mathrm{Cay}(G, S)$. The cocompactness of the action on G implies the following key geometric fact:

Proposition 3.7. *Let $\Gamma(G, \mathcal{Q})$ be a coned-off Cayley graph for G with respect to some finite generating set. The action of G on any hyperbolic graph Γ where*

(1) *G acts cocompactly by isometries,*
(2) *the infinite vertex stabilizers are maximal parabolics,*
(3) *the edge stabilizers are finite, and*
(4) *every maximal parabolic stabilizes a vertex*

induces a G-equivariant quasi-isometry $\Gamma(G, \mathcal{Q}) \to \Gamma$.

In particular, $X^{(1)}$ is a Gromov hyperbolic graph.

Another crucial property of small cancellation free products of non-elementary relatively hyperbolic groups is that they posses several relatively hyperbolic structures. Most importantly, these different structures are naturally witnessed by actions of the group on certain spaces associated to the decompositions as complexes of groups described in Section 3.1 and Section 3.4. An easy consequence of Proposition 3.6 is the following:

Proposition 3.8. *If $(A, \mathcal{P}_A)$ are $(B, \mathcal{P}_B)$ are relatively hyperbolic pairs and $\mathcal{P} = \mathcal{P}_A \cup \mathcal{P}_B$, then $(G, \mathcal{P})$ is a relatively hyperbolic pair.*

Notation 3.9. *For the remainder of this paper, we will set $\mathcal{Q} = \{A, B\}$ so that $(G, \mathcal{Q})$ is the relatively hyperbolic structure of G from Proposition 3.6 where the peripheral subgroups are entire vertex groups with the understanding that everything generalizes to any finite collection of factor groups as in Remark 3.5.*

We will use $(G, \mathcal{P})$ to denote the relatively hyperbolic structure with $\mathcal{P} = \mathcal{P}_A \cup \mathcal{P}_B$ coming from the peripherals of the vertex groups as in Proposition 3.8. Likewise, we will use $\partial_{\mathcal{Q}} G$ and $\partial_{\mathcal{P}} G$ to denote the respective Bowditch boundaries of $(G, \mathcal{Q})$ and $(G, \mathcal{P})$.

3.3. **Acylindricity of relative hyperbolic group actions.** Acylindrical actions on general hyperbolic spaces were defined by Bowditch [Bow08] extending prior notions due to Sela [Sel97] and Delzant [Del99]. The class of *acylindrically hyperbolic groups* is quite broad and natural. For example, relatively hyperbolic groups provide a rich source of acylindrically hyperbolic groups (see [Osi16] and references therin for more on acylindrical hyperbolicity). As will be apparent in Proposition 6.10, our methods only require controlling orbits of certain quasiconvex subsets of complex formed by subdividing X. We will make use of the following formulation of acylindrical actions introduced by Abbott and Manning [AM24, Definition 3.1].

Definition 3.10. *Let K act by isometries on a hyperbolic metric space Y, and let H be a subgroup that is quasi-isometrically embedded by the action. The action of K on Y is **acylindrical along** H if for some (equivalently any) H–cobounded subspace $W \subset Y$ the following condition holds:*

For any $\varepsilon \geq 0$, there exists constants $D = D(\varepsilon, W)$ and $N = N(\varepsilon, W)$ such that any collection of distinct cosets $\{g_1 H, \ldots, g_k H\}$ such that

$$\# \left\{ n : \operatorname{diam} \left(\bigcap_{i=1}^{k} \mathcal{N}_{\varepsilon}(g_i W) \right) > D \right\} \leq N.$$

*Groups K that admit isometric actions on non-elementary hyperbolic spaces that are acylindrical along the entire group are called **acylindrically hyperbolic**.*

In [AM24, Theorem 3.2], Abbott and Manning show that the notion of acylindricity in Definition 3.10 agrees with the notion defined by Bowditch.

In the setting of $C'(\frac{1}{6})$–small cancellation free products, acylindricity of the G action on X will be established in Proposition 6.8.
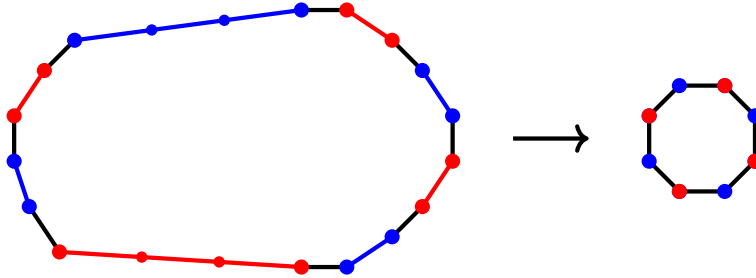
FIGURE 2. A blown-up polygon in $\mathcal{E}G$ and its image in the polygonal complex X

3.4. **Small cancellation blow-up.** The geometry of the factor groups is not reflected in the developed complex X. We will see how to enlarge X to a new complex $\mathcal{E}G$ that captures the cubical geometry of the factor groups using a construction of Martin and Steenbock [MS17, Definition 2.12] (see also prior work of Martin [Mar14, Definition 2.2]).

Let EA and EB denote CAT(0) cube complexes on which A and B respectively act cocompactly. Originally, Martin and Steenbock require that the actions on EA and on EB are also proper in order to conclude that the resulting blown-up space $\mathcal{E}G$ admits a geometric action by G. Note that the proof of [MS17, Proposition 2.14] only uses properness of these actions to show that the action of G on $\mathcal{E}G$ is again proper. In particular, when A and B only admit relatively geometric actions on EA and EB respectively, the action of G on $\mathcal{E}G$ is still cocompact, which we record below in Proposition 3.12.

We summarize key aspects of this construction to establish notation and for completeness. The reader should think of $\mathcal{E}G$ as replacing vertices of X with copies of either EA or EB in accordance with the local group. In doing this, verticies of polygons are blown up to geodesic segments within the vertex space as in Figure 2. For precise details about the attaching maps that ensure G still acts isometrically on this new space see [MS17, Definitions 2.16-2.18]. For our purposes, it is enough to know the existence of a projection $p : \mathcal{E}G \to X$. The geometry of $\mathcal{E}G$ is largely determined by the fibers over the vertices of X. These fibers are:

$$EG_x \coloneqq p^{-1}(x) \cong \begin{cases} EA & \text{when } G_x = A^g \text{ for some } g \in G \\ EB & \text{when } G_x = B^g \text{ for some } g \in G \\ \text{a single point} & \text{otherwise.} \end{cases}$$

The interior of each edge and polygon of X lifts uniquely to an edge or polygon of $\mathcal{E}G$. We will refer to EG_x as a **fiber complex** and G_x, its stabilizer, as a **fiber group**. We call the complex $\mathcal{E}G$ the **blow-up** of the developed complex X with respect to the actions of A and B on EA and EB.

Here is a very simple example of this construction:

Example 3.11. *Let $A = \mathbb{Z}$, let $B = \mathbb{Z}$ and let $G = A * B$. Let EA and EB be the standard Cayley graphs for the presentations $\mathbb{Z} \cong \langle a_0 \rangle$ and $\mathbb{Z} \cong \langle b_0 \rangle$ respectively. The complex of groups in this case is*

the standard graph of groups presentation for $G = F_2$ and its development is the standard Bass-Serre tree with distinguished vertices v_A and v_B stabilized by A and B respectively. Note that G does not act properly on the Bass-Serre tree T. In this case, the blow-up $\mathcal{E}G$ will be a $3-$valent tree on which G acts properly.

The space $\mathcal{E}G$ is constructed is as follows: pick distinguished vertices x_A and x_B in EA and EB. For each vertex v of T, let $\mathcal{E}G_v$ be a copy of either EA or EB depending on whether v is in the orbit of the vertex of T stabilized by A or in the orbit of the vertex stabilized by B. Let x_v be the copy of x_A or x_B as appropriate. We now glue together these spaces as follows: for each $a \in A$, there is an edge in T from v_A to $a \cdot v_B$ we connect the vertex $a \cdot x_A \in EA$ to the vertex $x_{a \cdot v_B} \in EG_{a \cdot v_B}$ by an edge. Likewise, for each $b \in B$, connect the vertex $b \cdot x_B \in EB$ to $x_{b \cdot x_A} \in EG_{b \cdot x_A}$ (except there should only be one edge between x_A and x_B). We can then extend these choices equivariantly to obtain a $3-$valent tree.
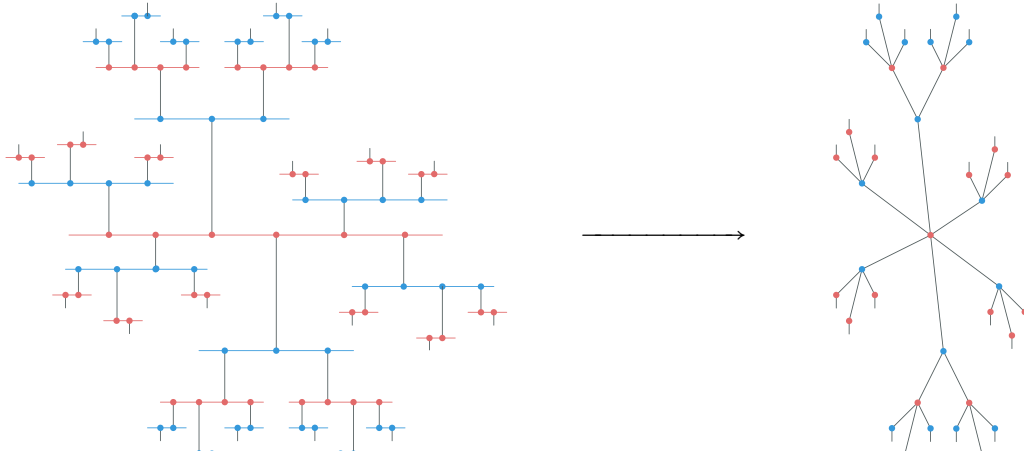

FIGURE 3. The blow-up described in Example 3.11 is the standard tree of spaces over the Bass-Serre tree

In general, we will only have a relatively geometric action on the fiber complexes, so the resulting action on $\mathcal{E}G$ will not be proper.

Using parts of [Mar14, Theorem 2.4] that do not require properness, we obtain the following:

Proposition 3.12. *The action of G on $\mathcal{E}G$ is cocompact, and $\mathcal{E}G$ is simply connected.*

Note that the properness hypotheses of [Mar14, Theorem 2.4] are not used in the cocompactness and contractibility arguments.

4. The Construction of the Walls

In this section, we review some of the objects Martin and Steenbock construct to help build their wallspace structure on $\mathcal{E}G_{bal}$, a subdivision of $\mathcal{E}G$ that is defined in Section 4.2.

We formalize **hyperstructures**, which are used implicitly in [MS17] (see Definition 4.2). Hyperstructures are inspired by hyperplanes in cube complexes. Similar ideas have been used to construct wallspaces for cubulation, see for example [Wis04]. Like hyperplanes, they have hypercarriers and dual edges. We will also encounter **hypergraphs** in X_{bal}, a subdivision of X that makes the projection $\mathcal{E}G_{bal} \to X_{bal}$ a combinatorial map. Hypergraphs are either hyperstructures of X_{bal} or projections of hyperstructures of $\mathcal{E}G_{bal}$ to X_{bal}. The notation in this section is based on [MS17] to help the reader refer between our work and theirs. Superscripts are intended to indicate which type of hyperstructure an object arises from, and subscripts indicate a dual edge that determines the hyperstructure an object arises from.

4.1. Ω–**hyperstructures.** Let Ω be a complex whose cells are polygons with an even number of edges. We can place an equivalence relation $\sim_{opp}$ on the edges of Ω as follows: for all edges e, $e \sim e$ and if U is a polygon in Ω and e_1, e_2 are diametrically opposite edges of the polygon U, then $e_1 \sim_{opp} e_2$. Then $\sim_{opp}$ can be extended to an equivalence relation on Ω by taking the transitive closure.

Example 4.1. *If Ω is a square complex, $e_1 \sim_{opp} e_2$ if and only if they are dual to the same hyperplane.*

More generally, if Ω is a complex where every cell is either a polygon with an even number of edges or an n-cube, we can place a similar relation on Ω: let U be a polygon or n-cube of Ω where $n \geq 2$. If e_1, e_2 are edges in U, we say $e_1 \sim_{opp} e_2$ when:

(1) if $e_1 = e_2$, then $e_1 \sim_{opp} e_2$

(2) if U is a polygon, then $e_1 \sim_{opp} e_2$ when they are diametrically opposed in P, and

(3) if U is an $n \geq 2$ cube, then $e_1 \sim_{opp} e_2$ when they are dual to the same midcube.

As before, $\sim_{opp}$ extends to an equivalence relation by taking the transitive closure.

Definition 4.2. *Let e be an edge in Ω. The Ω–**hyperstructure** associated to e, W_e^Ω, is the subspace of Ω constructed as follows: for each pair $e_1, e_2 \in [e]_{\sim_{opp}}$ where e_1, e_2 are edges of an $(n \geq 2)$–cell U,*

(1) *when U is a polygon, add straight segments joining the center of U to the midpoints of e_1 and e_2 or*

(2) *when e_1 and e_2 are dual to a midcube of an n-cube U in which case, include the midcube of U that is dual to e_1 and e_2 in W_e^Ω.*

The **carrier** of the hyperstructure W_e^Ω denoted Y_e^Ω is the union of all cells whose interior intersects W_e^Ω.

For example, if Ω is any cube complex, the Ω–hyperstructure associated to an edge e is the hyperplane dual to e.

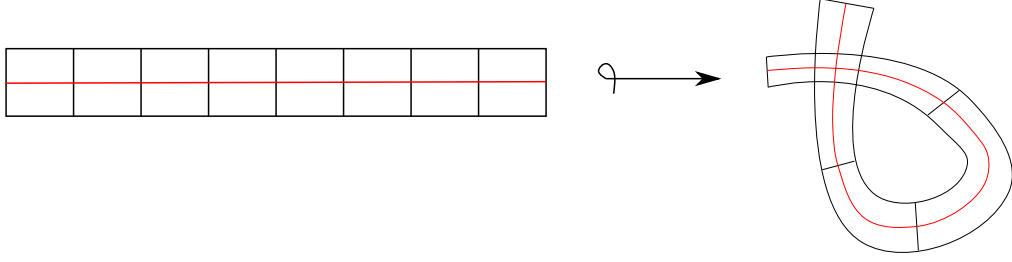The **abstract carrier of** W_e^Ω, is the complex $\widetilde{Y}_e^\Omega$ constructed as follows:

FIGURE 4. On the right is a square complex with a hyperplane drawn in red. On the left is its abstract carrier.

(1) For each pair of diametrically opposed edges of a polygon U in $[e]_{\sim_{opp}}$ include a copy of U with the edges e_1, e_2 labeled and the remaining edges unlabeled.

(2) For each collection of edges in $[e]_{\sim_{opp}}$ dual to the same midcube M of a maximal cube C, include a copy of C with copies of the edges dual to M labeled and the remaining edges unlabeled.

(3) If F, F' are cells of Y_e^Ω with labeled edges e, e' respectively so that e, e' have the same image $e_{F,F'} \in [e]_{opp}$, let E, E' be the maximal subcomplexes of F, F' respectively whose images in Ω agree and contain $e_{F,F'}$. Then glue F, F' in Y_e^Ω by identifying points in E, E' that have the same image in Ω.

We remark that when polygons in Ω have connected intersections (for example as a consequence of a small cancellation condition like Definition 2.9) then in Item 3 the paths ρ and ρ' can be chosen to be maximal paths whose images necessarily contain an edge $e_{U,U'} \in [e]_{\sim_{opp}}$. Note also that there is a natural surjective immersion $\widetilde{Y}_e^\Omega \to Y_e^\Omega$ from the abstract carrier to the carrier.

Example 4.3. *In Figure 4, we show a square complex with one of its hyperplanes and its abstract carrier. In this case, the abstract carrier immerses but does not embed.*

The following lemma will be prove useful later:

Lemma 4.4. *Let Ω be a complex where every cell is a polygon with an even number of cells or an n–cube for some n. Suppose that K acts combinatorially and cocompactly on Ω. If W_e^Ω is an Ω–hyperstructure, then $\mathrm{Stab}_K(W_e^\Omega)$ acts cocompactly on W_e^Ω.*

Proof. By construction hyperstructures are closed. Subsets of the quotient are closed if and only if their preimage is closed. Thus, the image of W_e^Ω is compact. Cocompactness follows because the the restriction of the quotient map to W_e^Ω is induced by the action of $\mathrm{Stab}_K(W_e^\Omega)$.   □

4.2. **Hypergraphs coming from the subdivided complex X_{bal}.**   We need to subdivide X and $\mathcal{E}G$ to obtain the appropriate walls. Specifically, we need to ensure that hyperstructures cannot enter and exit a single polygon too close together.

Definition 4.5 ([MS17, Definition 3.3]). *Let U be a polygon of a $C'(\frac{1}{6})$-polygonal complex X and τ_1, τ_2 be two simplices of the boundary ∂U. We say that τ_1, τ_2 are **far apart** in U if no path α in ∂U containing both τ_1 and τ_2 is a concatenation of strictly less than four pieces.*

Note that the notion of piece in Definition 4.5 is [MS17, Definition 2.9], which defines a *piece* of a polygonal complex as an injective path ρ contained in the boundary of two polygons, whose boundaries cannot be identified homeomorphically while keeping ρ fixed pointwise.

Recall from the construction of $\mathcal{E}G$ that the projection $p : \mathcal{E}G \to X$ is injective on edges whose interior is disjoint from every fiber spaces. These edges are referred to as **horizontal edges** of the blow-up $\mathcal{E}G$. Edges whose interiors are contained in a fiber space are called **vertical edges** (see also [MS17, Definition 2.17])

We subdivide $\mathcal{E}G$ in two steps with respect to an integer $k \geq 0$. First, subdivide every vertical edge once and then cubically subdivide the fiber complexes to match the vertical subdivisions. This guarantees that polygons are glued to fiber complexes along paths of even length. Second, subdivide each horizontal edge into k edges. The projection $p : \mathcal{E}G \to X$ will remain a combinatorial projection when every edge of X is subdivided into k edges.

The inequality in Definition 2.9 is strict, so for all k sufficiently large the resulting subdivision $\mathcal{E}G_{bal}$ of $\mathcal{E}G$ and subdivision X_{bal} of X have the following two properties:

  (1) If U is a polygon of $\mathcal{E}G_{bal}$ or X_{bal}, then U has an even number of sides,
  (2) Under the natural projection $p : \mathcal{E}G_{bal} \to X_{bal}$, every pair of diametrically opposed edges of a polygon U in $\mathcal{E}G_{bal}$ projects to a pair of edges that are far apart in $p(U)$ in X_{bal} [MS17, Lemma 3.24].

The subdivisions $\mathcal{E}G_{bal}$ and X_{bal} are called **balanced subdivisions**.

For each edge e of X_{bal} we define Z_e^X to be the X_{bal}-hyperstructure associated to e which we refer to as the **hypergraph of X_{bal} associated to** e. The **carrier** or **hypercarrier** Y_e^X of Z_e^X is the smallest subcomplex of X_{bal} that contains Z_e^X.

Rather than constructing a unified wallspace structure in $\mathcal{E}G_{bal}$ like Martin and Steenbock do, we will construct a sufficient family of codimension-1 subgroups that satisfy the hypotheses of Theorem 2.7 from the hyperstructures of *both* X_{bal} and $\mathcal{E}G_{bal}$.

4.3. **The hyperstructures of $\mathcal{E}G_{bal}$.**   In [EGN24], we prove that any pair of distinct points in the Bowditch Boundary of a relatively geometrically cubulated group are separated by the limit set of a hyperplane stabilizer. An intuitive way to try to separate points in the Bowditch Boundary of a single fiber group is to use the hyperplanes of the fiber complexes, but we need to extend the hyperplanes to ensure that they separate $\mathcal{E}G$.

If e is an edge of a fiber complex of $\mathcal{E}G_{bal}$, we define $W_e^{\mathcal{E}G}$ to be the $\mathcal{E}G_{bal}$–hyperstructure associated to e, which is two-sided and contractible by [MS17, Lemma 3.34].

In the specific case of $(G, \mathcal{P})$, the codimension–1 subgroups that we use to separate points in $\partial_{\mathcal{P}} G$ that lie in the Bowditch Boundary of a single fiber group will be finite index subgroups of the $\mathrm{Stab}_G(W_e^{\mathcal{E}G})$.

The far apart condition will ensure that projections of $W_e^{\mathcal{E}G}$ have properties similar to the X_{bal}–hyperstructures. We now introduce a few facts about these **projected hypergraphs**.

Definition 4.6. *Let $W_e^{\mathcal{E}G}$ be an $\mathcal{E}G_{bal}$–hyperstructure. The **projected hypergraph** $Z_e^{\mathcal{E}G}$ is the image of $W_e^{\mathcal{E}G}$ under the projection $p : \mathcal{E}G_{bal} \to X_{bal}$.*

Unlike the hypergraphs Z_e^X associated to classes of opposite edges in X_{bal}, it is possible that the intersection of a projected hypergraph $Z_e^{\mathcal{E}G}$ with the boundary of a polygon in X_{bal} includes a vertex.

Definition 4.7. *The **hypercarrier** $V_e^{\mathcal{E}G}$ **associated to** $Z_e^{\mathcal{E}G}$ is the union (in X_{bal}) of the polygons of X_{bal} whose interior intersects $Z_e^{\mathcal{E}G}$.*

We can also construct an **abstract hypercarrier** $\widetilde{V}_e^{\mathcal{E}G}$ **associated to** $Z_e^{\mathcal{E}G}$ as follows: let $\widetilde{Y}_e^{\mathcal{E}G}$ be the abstract carrier obtained by viewing $W_e^{\mathcal{E}G}$ as an $\mathcal{E}G_{bal}$–hyperstructure. For each polygon $\widehat{U}$ of $\widehat{Y}_e^{\mathcal{E}G}$, let $q(\widehat{U})$ be the image in X_{bal} after mapping $\widehat{U}$ into $\mathcal{E}G_{bal}$ and applying the projection $p : \mathcal{E}G_{bal} \to X_{bal}$. Note that $q(\widehat{U})$ is a polygon of X_{bal} (but may have fewer boundary edges than $\widehat{U}$). For each $\widehat{U}$, $\widehat{V}_e^{\mathcal{E}G}$ contains a copy of $q(\widehat{U})$. If ρ is common piece of $\widehat{U_1}$ and $\widehat{U_2}$, the copies of $q(\widehat{U_1})$ and $q(\widehat{U_2})$ are glued together along the image of $q(\rho)$ which may be either a path or a vertex.

Martin and Steenbock's argument showing that the natural immersion of a hypercarrier of W_e^X is an embedding also applies to the natural immersion $\widetilde{V}_e^{\mathcal{E}G} \to V_e^{\mathcal{E}G}$:

Proposition 4.8 ([MS17, Theorem 3.10]). *The natural immersion $\widetilde{V}_e^{\mathcal{E}G} \to V_e^{\mathcal{E}G}$ is an embedding.*

The equivariance of the projection $\mathcal{E}G_{bal} \to X_{bal}$ also implies that $\mathrm{Stab}_G(W_e^{\mathcal{E}G}) \leq \mathrm{Stab}_G(Z_e^{\mathcal{E}G})$.

In the following, we adopt the convention that sets $Z = Z_e^X$ and $Z = Z_e^{\mathcal{E}G}$ are subsets of X_{bal} and are called **hypergraphs (of X_{bal})**.

Example 4.9. *We briefly recall Example 3.11 to give very basic examples of these objects. For an edge e in EG_v, $W_e^{\mathcal{E}G}$ is the midpoint of e. Its carrier is e. Likewise for an edge f in the Bass-Serre tree, Z_f^X is the midpoint of the edge f. The carrier of Z_f^X is f.*

5. Realizing and working with the Bowditch Boundary

In this section, we build tools that will help us turn stabilizers of hypergraphs in X_{bal} and $\mathcal{E}G$–hyperstructures into codimension–1 subgroups that separate points in the Bowditch boundary. Our ultimate goal is to use these subgroups for Theorem 2.7. A given group K may admit multiple relatively hyperbolic structures, which yield different Bowditch boundaries. In Section 5.1, we prove that we can often separate points in the Bowditch boundary of a refined peripheral structure by

separating their preimages in the boundary of the original structure. The remainder of the section, especially Theorem 5.15, adapts prior work of the authors with Groves [EGN24] on separating pairs of points in Bowditch boundaries with quasiconvex subgroups.

5.1. **Separating Points in the Bowditch Boundary of a Refined Peripheral Structure.**
Here we consider $(K, \mathcal{R})$, a refined peripheral structure for a relatively hyperbolic pair $(K, \mathcal{D})$ where each $R \in \mathcal{R}$ is conjugate into some $D \in \mathcal{D}$. In our application, $(K, \mathcal{D})$ will be derived from a blowup construction modeled after $\mathcal{E}G_{bal}$ that we introduce in Section 8. In this subsection, we lay the groundwork so that in Section 8, we can apply Theorem 2.7, the relatively geometric cubulation criterion, if we can show that:

(1) any two points in the Bowditch Boundary of $(K, \mathcal{D})$ can be separated by the limit set of a full relatively quasiconvex (in $\mathcal{D}$) codimension–1 subgroup, and

(2) any two points in the Bowditch Boundary of $(K, \mathcal{R})$ that lie in the limit set of D^g can be separated by a full relatively quasiconvex codimension–1 subgroup.

The coarser relative structure $(K, \mathcal{D})$ is a sometimes also called a *parabolically extended structure* for $(K, \mathcal{R})$ [Yan14]. Yang proves the following useful relationship between their Bowditch boundaries:

Proposition 5.1 ([Yan14, Lemmas 4.13-4.14]). *There exists a continuous K–equivariant surjective map $\phi : \partial_{\mathcal{R}} K \to \partial_{\mathcal{D}} K$. Further, for any conical limit point x of $\partial_{\mathcal{D}} K$, $\phi^{-1}(x)$ consists of a single point.*

In particular, the map ϕ collapses the entire limit set $\Lambda_{\mathcal{R}} D^g$ of a maximal parabolic in $(G, \mathcal{R})$ to the parabolic point it stabilizes in $\partial_{\mathcal{D}} K$. Separating points in the Bowditch Boundary is sometimes more straightforward in one relative structure than another.

Proposition 5.2. *Let $H \leq G$ be a relatively quasiconvex in $(K, \mathcal{D})$ and let $\Lambda_{\mathcal{D}} H$ and $\Lambda_{\mathcal{R}} H$ be the limit sets of Λ in $(K, \mathcal{D})$ and $(K, \mathcal{R})$ respectively.*

If x and y lie in distinct H–components of $\partial_{\mathcal{D}} K \smallsetminus \Lambda_{\mathcal{D}} H$, then any $x_0 \in \phi^{-1}(x)$ and $y_0 \in \phi^{-1}(y)$ lie in distinct H–components of $\partial_{\mathcal{R}} K \smallsetminus \Lambda_{\mathcal{R}} H$.

Proof. Since ϕ is continuous and equivariant, if x_0 and y_0 were in the same H–component of $\partial_{\mathcal{D}'} K \smallsetminus \Lambda_{\mathcal{D}'} H$, their images under ϕ must lie in the same H–component of $\partial_{\mathcal{D}} K \smallsetminus \Lambda_{\mathcal{D}} H$. $\qquad \square$

5.2. **Fine hyperbolic graphs, generalized fine actions and the Bowditch Boundary.**
Later, via a more general framework, we will see that $X_{bal}^{(1)}$ is a fine hyperbolic graph that can be used to realize the Bowditch Boundary of $(G, \mathcal{Q})$. Similarly, $\mathcal{E}G_{bal}^{(1)}$ can be used to realize the Bowditch Boundary of $(G, \mathcal{P})$. In this subsection, we introduce some technical details about the relationship between fine hyperbolic graphs, generalized fine actions defined below and the Bowditch Boundary of a relatively hyperbolic group.

Suppose that $(K, \mathcal{D})$ acts relatively geometrically on a CAT(0) cube complex $\widetilde{C}$. Then [CC07, Theorem 5.1] implies that both $\widetilde{C}$ and $\widetilde{C}^{(1)}$ are quasi-isometric to the coned-off Cayley graph. Note that $\widetilde{C}^{(1)}$ may have infinite edge stabilizers and therefore fails to be a fine hyperbolic graph with finitely many K-orbits of edges and finite edge stabilizers.

Definition 5.3. *Let Γ be a graph and let $\mathcal{B}$ be a collection of pairwise disjoint connected sub-graphs of Γ. The **complete electrification of** Γ **with respect to** $\mathcal{B}$ is the graph Γ' constructed by collapsing each $B \in \mathcal{B}$ to a single vertex v_B.*

Recent work of the authors and Groves [EGN24] introduced the following notion in which complete electrifications relate a graph like $\widetilde{C}^{(1)}$ to a fine hyperbolic graph with compact fundamental domain and finite edge stabilizers.

Definition 5.4. *Let K be a group that acts by isometries on a δ–hyperbolic cell complex, and let $\mathcal{D}$ be a finite and almost malnormal collection of finitely generated subgroups of K. For each $D \in \mathcal{D}$ and $k \in K$, let Σ_{D^k} be the sub-graph consisting of cells whose stabilizers are **commensurable** to D^k; that is, those cells whose stabilizer is finite index in D^k. A **circuit without peripheral backtracking** is a circuit whose intersection with any Σ_{D^k} is connected. We say that Σ is **generalized fine** with respect to the action of $(K, \mathcal{D})$ if:*

(1) *the quotient $_K\backslash^{\Sigma}$ is compact,*

(2) *every cell with infinite stabilizer lies in some Σ_{D^k},*

(3) *the sub-graphs Σ_{D^k} are compact and connected, and*

(4) *for every edge e of Σ with finite stabilizer and every $n \in \mathbb{N}$, there are finitely many circuits without peripheral backtracking of length n that contain e.*

Collapsing the stabilized sub-graphs Σ_{D^k} yields a fine hyperbolic graph. We will see that $\mathcal{E}G_{bal}^{(1)}$ is generalized fine with respect to the action of $(G, \mathcal{P})$. We now introduce some properties of generalized fine actions from [EGN24, Proposition 3.4 and Proposition 3.5]:

Proposition 5.5. *Let Σ be generalized fine with respect to the action of $(K, \mathcal{D})$. The graph Σ' obtained by electrifying $\mathcal{B} = \{\Sigma_{D^k} : D \in \mathcal{D}, k \in K\}$ is a fine hyperbolic graph with a K-cocompact action where all edge stabilizers are finite. Moreover, the collapse map $\beta : \Sigma \to \Sigma'$ that collapses the Σ_{D^k} is a K-equivariant quasi-isometry.*

Bowditch explains how to construct $\partial_{\mathcal{D}} K$ from a suitable action on a fine hyperbolic graph [Bow12, Section 9]. We recall this construction and how to extend it to generalized fine actions as in [EGN24, Definition 4.6]:

Definition 5.6. *Let $(K, \mathcal{D})$ be a relatively hyperbolic pair and suppose that Γ is a graph that is generalized fine with respect to the action of $(K, \mathcal{D})$. For each $D \in \mathcal{D}$ and $k \in K$, let Γ_{D^k} be the sub-graph consisting of cells whose stabilizers are commensurable to D^k. Let Γ' be a complete*

electrification of Γ *with respect to the* Γ_{D^k} *and for all* $D \in \mathcal{D}$ *and* $k \in K$, *let* v_{D^k} *be the vertex of* Γ' *stabilized by* D^k. *Let* $\triangle\Gamma' = \partial\Gamma' \sqcup V(\Gamma')$ *endowed with the following topology: If* A *is any finite subset of the vertices of* Γ' *and* $a \in \triangle\Gamma'$, *define* $N(a, A)$ *to be the set of* $b \in \triangle\Gamma'$ *so that every geodesic from* a *to* b *avoids* $A \setminus a$. *A subset* $U \subseteq \triangle\Gamma'$ *is open if for every* $a \in U$, *there exists a finite set of vertices* $A \subseteq V(\Gamma')$ *so that* $N(a, A) \subseteq U$.

Let $\Pi_{\Gamma'} = \{v_{D^k} : D \in \mathcal{D}, k \in K\}$.

The **Bowditch Boundary** *of* $(K, \mathcal{D})$ *is* $\partial_{\mathcal{D}}K = \partial\Gamma' \cup \Pi_{\Gamma'}$ *where* $\partial\Gamma'$ *is the visual boundary of the hyperbolic graph* Γ'. *The topology on* $\partial_{\mathcal{D}}K$ *is the subspace topology induced by the topology on* $\triangle\Gamma'$.

5.3. **Hypersets in Graphs.**   Let Γ be a connected graph. A **hyperset** L in Γ is a collection of edge midpoints and vertices of Γ so that $\Gamma \setminus L$ has two components. We will be mostly concerned with sets that arise as intersections of the walls from Section 4 with the one-skeleton of their ambient space. A **hyperset carrier** J is a minimal connected induced sub-graph of Γ containing L having the property that the intersection of J with each complementary component of $\Gamma \setminus L$ is connected.

Here are the two natural ways that hypersets and carriers will arise for us:

Example 5.7. *Let* $(K, \mathcal{D})$ *act relatively geometrically on a CAT(0) cube complex* $\widetilde{C}$, *and let* W *be a hyperplane. Then* $L = W \cap \widetilde{C}^{(1)}$ *is a hyperset. A hypercarrier* J *for* L *is the intersection of the carrier of* W *with* $\widetilde{C}^{(1)}$.

In this situation, we will refer to L as the **hyperset associated to** W and J as **the hypercarrier** of the hyperset associated to W.

Example 5.8. *Suppose* Ω *is a complex consisting of polygons with an even number of sides and cubes. Let* W *be an* Ω–*hyperstructure that separates* Ω *into two complementary components, or if* $\Omega = X_{bal}$ *as defined above, then* W *can be also be a projected hypergraph. Then* $L = W \cap \Omega^{(1)}$ *is a hyperset. A hypercarrier* J *for* L *is the intersection of the associated hypercarrier for* W *with* Ω.

As in the cube complex case, we will refer to L as the **hyperset associated to** W and J as **the hypercarrier** of the hyperset associated to W. Let Σ be a δ-hyperbolic graph that is generalized fine with respect to the action of $(K, \mathcal{D})$ and for each $D \in \mathcal{D}$ and $k \in K$, let Σ_{D^k} be the compact sub-graph stabilized by D^k.

Hypersets are preserved when electrifying stabilized sub-graphs of a generalized fine action.

Proposition 5.9. *Let* Σ *be generalized fine with respect to the action of* $(K, \mathcal{D})$ *and let* Σ_{D^k} *be the sub-graph of cells whose stabilizer is commensurable to* D^k *for* $D \in \mathcal{D}$, $k \in K$. *Let* $\sigma : \Sigma \to \Gamma$ *be the map that collapses the* Σ_{D^k}. *Let* $S = \bigcup\{\Sigma_{D^k} : \Sigma_{D^k} \cap L \neq \varnothing\}$. *If* L *is a hyperset and* J *is a quasiconvex hyperset carrier, then* $\sigma(L)$ *is a hyperset in* $\Sigma(D^k)$, $\sigma(J)$ *is a hyperset carrier and the components of* $\Gamma \setminus \sigma(L)$ *are images of the components of* $\Gamma \setminus (S \cup L)$.

5.4. **A Separation Criterion for the Bowditch Boundary.** In this section, we recall a criterion from [EGN24] for separating points in the Bowditch Boundary constructed from a fine hyperbolic graph.

We set the following assumptions for the remainder of this subsection:

Hypotheses 5.10. *Let $(K, \mathcal{D})$ be a relatively hyperbolic pair and let K act on a fine δ–hyperbolic graph Γ with the following properties:*

- *The action of K is cocompact,*
- *edge stabilizers are finite, and*
- *each $D \in \mathcal{D}$ stabilizes a single vertex.*

Let L be a hyperset with connected quasiconvex carrier J.

Definition 5.11. *With the setup in Hypotheses 5.10, J has the **two-sided carrier property** if there exist connected quasiconvex subsets J^+ and J^- so that $J = J^+ \cup J^-$, $J^+ \cap J^- \subseteq L$, every path between vertices in distinct components of $\Gamma \smallsetminus L$ must intersect both J^+ and J^- and if v is a vertex of $J^- \cap J^+$ with infinite stabilizer, then $v \in \Lambda \operatorname{Stab}_K(L)$.*

When Hypotheses 5.10 arise from electrifying the stabilized sub-graphs of $(K, \mathcal{D})$ acting relatively geometrically on a CAT(0) cube complex $\widetilde{X}$, the two-sidedness of the hyperplanes gives us a natural way to show that hypersets in Γ have the two-sided carrier property, see [EGN24, Proposition 6.3].

When the Bowditch Boundary is constructed as in Definition 5.6 from a fine hyperbolic graph, the points of the Bowditch Boundary are either **conical limit points** that lie in $\partial \Gamma$, the visual boundary of Γ or are **parabolic vertices** of Γ that are stabilized by maximal parabolics.

Note that Γ is not a proper metric space, so existence of a bi-infinite geodesic connecting points in the visual boundary of $X_{bal}^{(1)}$ does not follow immediately from the standard techniques for constructing such a bi-infinite geodesic.

Proposition 5.12. *Let $x, y \in \partial_{\mathcal{D}} K$. There exists a bi-infinite combinatorial geodesic between x and y in Γ.*

Proof. There is a proper hyperbolic space Z so that $\partial Z \cong \partial_{\mathcal{D}} K$. Let $\triangle \Gamma = \Gamma^{(1)} \cup \partial \Gamma$. By [Bow12, Lemma 9.1], there exists a map from $\gamma : \triangle \Gamma \to Z$ that induces a homeomorphism $\overline{\gamma} : \partial_{\mathcal{D}} K \to \partial Z$. Since Z is proper and hyperbolic, there exists a bi-infinite geodesic α in Z that joins $\overline{\gamma}(x)$ and $\overline{\gamma}(y)$. By [Bow12, Lemma 9.3], there exists a bi-infinite geodesic Υ in Γ so that $\gamma(\Upsilon)$ lies a bounded distance from α, so Υ joins x to y in $\partial_{\mathcal{D}} K$. $\qquad\square$

Definition 5.13. *Assuming Hypotheses 5.10, we say that L **separates** p, q if $p, q \notin \Lambda \operatorname{Stab}_K(L)$ and one of the following holds:*

- *p, q are both conical limit points, and there exists some geodesic $\gamma : (-\infty, \infty) \to \Gamma$ with $\lim_{t \to \infty} \gamma(t) = p$ and $\lim_{t \to -\infty} \gamma(t) = q$ so that there exists $T > 0$ so that for all $t_- < -T < 0 < T < t_+$, $\gamma(t_+)$ and $\gamma(t_-)$ are in distinct components of $\Gamma \smallsetminus L$,*

- q is a parabolic vertex in Γ, p is a conical limit point, and there exists some geodesic $\gamma :$ $[0, \infty) \to \Gamma$ from $q = \gamma(0)$ to p so that for t sufficiently large, $\gamma(t)$ and q are in distinct components of $\Gamma \smallsetminus L$,
- p, q are both parabolic vertices in Γ, and p, q lie in distinct components of $\Gamma \smallsetminus L$.

For the next theorem, we need a quick definition:

Definition 5.14. *Let G act on the components of a topological space and let $H \leq G$. Two components C_1, C_2 are H–distinct if $h \cdot C_1 \neq C_2$ for all $h \in H$.*

In [EGN24, Theorem 5.7], Groves and the authors prove that separation in the sense of Definition 5.13 implies separation in the Bowditch Boundary:

Theorem 5.15 ([EGN24, Theorem 5.7]). *Assume Hypotheses 5.10 and assume the setup satisfies the two-sided carrier property. If L separates $p, q \in \partial_{\mathcal{D}} K \smallsetminus \Lambda \operatorname{Stab}_K(L)$, then there exists a finite index subgroup $H_L \leq \operatorname{Stab}_K(L)$ so that p and q are in H_L–distinct components of $\partial_{\mathcal{D}} K \smallsetminus \Lambda H_L$.*

6. The Polygonal Case

6.1. Separating boundary points in a polygonal complex when the hyperstructures are trees. For Section 6.1, we set these hypotheses:

Hypotheses 6.1. *Let X_0 be a polygonal complex where each polygon has an even number of sides. Let $(K, \mathcal{D})$ be a relatively hyperbolic pair that acts cocompactly on X_0 so that every edge has finite stabilizer, every $D \in \mathcal{D}$ stabilizes exactly one vertex and every infinite cell stabilizer is of the form D^k for some $D \in \mathcal{D}$ and $k \in K$.*

As in the more specific case of when $X_0 = X_{bal}$, given an edge e of X_0, we let Z_e^X denote the X_0–hyperstructure dual to e. We also refer to Z_e^X as a hypergraph.

We now impose some additional conditions that we will subsequently prove are satisfied by X_{bal}:

Hypotheses 6.2. *In addition to Hypotheses 6.1, we assume that:*

(1) *For any edge e of X_0, Z_e^X is an embedded tree in X_0 whose intersection with $X_0^{(1)}$ is quasi-convex in $X_0^{(1)}$,*

(2) *For any edge e of X_0, Z_e^X separates X_0 into two complementary components.*

(3) *Further, $\operatorname{Stab}_K(Z_e^X)$ has finite intersection with any vertex stabilizer and is therefore full.*

(4) *If γ is a combinatorial geodesic, there exists an edge e of γ so that Z_e^X crosses γ exactly once. If γ is infinite, there exists an $N \in \mathbb{N}$ so that every subsegment of γ with length N contains an edge e so that Z_e^X intersects γ exactly once.*

If X_0 satisfies these properties, then we say X_0 has **suitable walls**.

We make a quick observation so we can use results from Section 5.2.

Proposition 6.3. *The graph $X_0^{(1)}$ is fine and hyperbolic.*

Proof. The graph $X_0^{(1)}$ is hyperbolic by Proposition 3.7. Since the edge stabilizers are finite, each maximal parabolic stabilizes exactly one vertex and the action of G is cocompact, so by [Bow12, Lemma 4.5], $X_0^{(1)}$ is fine.   □

Proposition 6.4. $\mathrm{Stab}_K(Z_e^X)$ *acts cocompactly on* Z_e^X, *and* $\mathrm{Stab}_K(Z_e^X)$ *is relatively quasiconvex in* $(K, \mathcal{D})$.

Proof. Cocompactness follows immediately from Lemma 4.4. Relative quasiconvexity then follows by Hypotheses 6.2(1) and Theorem 2.2.   □

If Z_e^X is a hypergraph in X_0, then the associated hyperset is $Z_e^X \cap X_0^{(1)}$. Similarly, if Y_e^X is the carrier of Z_e^X, then the associated hyperset carrier is $Y_e^X \cap X_0^{(1)}$, which we denote by J for ease of notation. We say that a hypergraph Z_e^X **separates** $p, q \in \partial_{\mathcal{D}} K$ if the associated hyperset L separates p, q in $\partial_{\mathcal{D}} K \smallsetminus \Lambda \mathrm{Stab}_K(Z_e^X)$ in the sense of Definition 5.13.

We recall some elementary facts about δ–hyperbolic spaces.

Lemma 6.5 (Uniform fellow-traveling). *Let X_0 be a δ–hyperbolic space. Let $U \subseteq X_0$ be any convex subset. There exists $R = R(\delta)$ such that if γ is a geodesic so that $\gamma \subseteq \mathcal{N}_\delta(U)$ then $\gamma \subseteq \mathcal{N}_R(U)$.*

Proof. This is an immediate consequence of the Morse lemma for hyperbolic spaces. We can view γ as a $(1, 2\delta)$–quasi-geodesic joining pairs of points in U.   □

Definition 6.6. *Let $\gamma \subseteq X_0^{(1)}$ be a (possibly finite) geodesic so that any endpoints are vertices. We define the **skewer set** as follows*

$$\mathrm{sk}(\gamma) = \{Z_e^X \mid e \text{ is an edge of } \gamma\}$$

We say that each $Z_e^X \in \mathrm{sk}(\gamma)$ is **dual** to γ.

Remark 6.7. *If γ is infinite and Hypotheses Hypotheses 6.2 hold, there are infinitely many edges in γ whose dual hypergraph crosses γ exactly once. Therefore, $\mathrm{sk}(\gamma)$ is infinite when γ is a geodesic ray or bi-infinite geodesic. There are finitely many K-orbits of edges in X_0. If γ is infinite, there is a hypergraph Z with infinitely many K-translates in $\mathrm{sk}(\gamma)$ that cross γ exactly once which we will denote:*

$$\mathcal{S}_\gamma := \{g_i Z\}_{i=0}^\infty.$$

Without loss of generality we can and will assume $g_0 = id_K$. Moreover, each hypergraph $O \in \mathrm{sk}(\gamma)$ disconnects γ into two components such that each lie in distinct components of $X_0^{(1)} \smallsetminus O$.

Our goal in describing skewer sets is that we may understand stabilizers of hypergraphs using acylindricity of the G–action on X_0 from the relative hyperbolicity of $(K, \mathcal{D})$:

Proposition 6.8. *Let X_0 and $(K, \mathcal{D})$ be as in Hypotheses 6.1. The action of K on $X_0^{(1)}$ is acylindrical.*

Proof. [CC07, Theorem 5.1] implies the orbit map induces a K–equivariant quasi-isometry from any relative Cayley graph $\mathrm{Cay}(K, \mathcal{D}, S)$ with S finite to $X_0^{(1)}$. The action of K on $\mathrm{Cay}(K, \mathcal{D}, S)$ is acylindrical [Osi16, Proposition 5.2].

Hence, the action of K on $X_0^{(1)}$ is acylindrical because acylindricity is preserved under equivariant quasi-isometry. $\qquad\square$

Note that in the definition of an action being acylindrical along a subgroup Definition 3.10 requires the subgroup H to be quasi-isometrically embedded in K with respect to any word metric associated to a finite generating set.

Lemma 6.9. *The orbit of $\mathrm{Stab}_K(Z_e^X)$ is quasi-isometrically embedded in $X_0^{(1)}$.*

Proof. By fullness, the intersection of $\mathrm{Stab}_K(Z_e^X)$ with any cell stabilizer of $X_0^{(1)}$ is finite. Therefore, $\mathrm{Stab}_K(Z_e^X)$ acts properly on the tree Z_e^X. By Proposition 6.4, $\mathrm{Stab}_K(Z_e^X)$ acts cocompactly on Z_e^X. Therefore, distance in $\mathrm{Stab}_K(Z_e^X)$ is coarsely equivalent to distance in Z_e^X, which is in turn coarsely equivalent to distance in $X_0^{(1)}$ because $Z_e^X \cap X_0^{(1)}$ is quasiconvex in $X_0^{(1)}$. $\qquad\square$

Recall that the Bowditch Boundary $\partial_{\mathcal{D}} K$ can be partitioned into parabolic points and conical limit points as in Definition 5.6. Moreover, since $X_0^{(1)}$ is a fine hyperbolic graph that witnesses the relative hyperbolicity of $(K, \mathcal{D})$, parabolic points can be identified with vertices in $X_0^{(1)}$ and conical limit points can be viewed as points in the visual boundary $\partial_{\infty} X_0^{(1)}$.

Proposition 6.10. *Let $p, q \in \partial_{\mathcal{D}} K$ be a pair of distinct points. Then there exists a hypergraph Z_e^X whose associated hyperset separates p and q.*

Proof. The partition of the Bowditch Boundary into parabolic and conical limit points splits our consideration into the following three cases: (1) both points are parabolic, (2) both points are conical limit points, and (3) one point is conical and the other is parabolic.

Case 1: p, q are both parabolic points.

Let γ be a geodesic segment in $X_0^{(1)}$ joining p and q. By Hypotheses 6.2(4), there exists an edge e_0 so that the dual hypergraph $Z_{e_0}^X$ crosses γ exactly once. Let $H = \mathrm{Stab}_K(Z_{e_0}^X)$.

In this case, let $Z_e^X = Z_{e_0}^X$. Since Z_e^X splits $X_0^{(1)} \smallsetminus Z_e^X$ into two components by Hypotheses 6.2(2), it suffices to show that $p, q \notin \Lambda H$. By Hypotheses 6.2(3) the intersection of the parabolic stabilizer of p or q with H is finite. Therefore, both p and q are not contained in the limit set of H.

Case 2: p, q are both conical limit points.

By Proposition 5.12 there is a geodesic γ in $X_0^{(1)}$ joining p, q. Let $\mathcal{S}_{\gamma}$ be the subset of $\mathrm{sk}(\gamma)$ described in Remark 6.7. Fix $R = R(\delta)$, the uniform fellow-traveling constant of Lemma 6.5. We claim that all but finitely many $Z_f^X \in \mathcal{S}_{\gamma}$ have $p \notin \Lambda \mathrm{Stab}_K(Z_f^X)$. Suppose not. Hypotheses 6.2(4)

implies that there are infinitely many edges whose dual hypergraph crosses γ exactly once, and there are finitely many edge orbits. Therefore we may choose $g_1, g_2, \ldots$ elements of K so that $g_1 e_0, g_2 e_0, \ldots$ is an infinite collection of edges of γ so that

(1) $Z_{g_i e_0}^X \in \mathcal{S}_\gamma$, and
(2) if $H_i = \operatorname{Stab}_K(Z_{g_i e_0}^X)$, then $p \in \Lambda H_i$.

The $Z_{g_i e_0}^X$ are distinct because they each cross γ exactly once. Therefore, $\{g_i H\}$ is an infinite collection of cosets of H because the orbit of $g_i H$ intersects $g_i e$. Parameterize $\gamma : (-\infty, \infty) \to X_0^{(1)}$ so that $\lim_{t \to \infty} \gamma(t) = p$. Then there exists t_i so that if $t > t_i$, $\gamma(t) \subseteq \mathcal{N}_R(Z_{g_i e_0}^X)$. Therefore for any $D \geq 0$:

$$\# \left\{ k : \operatorname{diam}\left(\bigcap_{i=0}^{k} \mathcal{N}_{2R}(Z_{g_i e_0}^X) \right) > D \right\}$$

is unbounded. Since $Z_{e_0}^X$ is H-cocompact, the image of $g_i H$ under the orbit map is coarsely $Z_{g_i e_0}^X$. Thus we obtain a contradiction to the fact that the action of K is acylindrical in the sense of Definition 3.10.

Similarly, all but finitely many $Z_f^X \in \mathcal{S}_\gamma$ do not have $q \in \Lambda \operatorname{Stab}_K(Z_f^X)$.

Therefore there exists some $Z_e^X \in \mathcal{S}_\gamma$ so that $p, q \notin \Lambda \operatorname{Stab}_K(Z_e^X)$. Since γ can only cross Z_e^X once, we see that p and q are separated by Z_e^X.

Case 3: p is a conical limit point and q is a parabolic vertex.

There exists a geodesic ray γ from $q = \gamma(0)$ to p by the definition of the visual boundary. By following the same argument in the previous case, there exists an e dual to γ so that $p \notin \Lambda \operatorname{Stab}_K(Z_e^X)$ and Z_e^X crosses γ exactly once. The same argument from the first case tells us that $q \notin \Lambda \operatorname{Stab}_K(Z_e^X)$. Since γ crosses Z_e^X exactly once and Z_e^X separates $X_0^{(1)}$ into two complementary components, Z_e^X separates p and q. $\qquad\square$

Proposition 6.11. *If J is the associated hyperset carrier of Z_e^X, then J has the two-sided carrier property.*

Proof. By Hypotheses 6.2(2), Z_e^X separates X_0 into two distinct components and therefore $L = Z_e^X \cap X_0^{(1)}$, the hyperset associated to Z_e^X, naturally separates $X_0^{(1)}$ and J decomposes into two distinct connected pieces J^+ and J^- so that $J^+ \cap J^- = L$ and $J^+ \smallsetminus L$, $J^- \smallsetminus L$ lie in distinct components of $X_0^{(1)} \smallsetminus L$. A combinatorial path from one component of $X_0 \smallsetminus Z_e^X$ must pass through an edge dual to Z_e^X, so every path between J^+ and J^- must pass through L. Recall that J is the convex one-skeleton of the carrier of Z_e^X and the cocompactness of the action of K on X_0 gives a bound M on the number of sides of any polygon in X_0, so J^+ and J^- are M–quasiconvex in $X_0^{(1)}$. By construction $J^+ \cap J^- = L$. $\qquad\square$

Theorem 6.12. *Let $p, q \in \partial_\mathcal{D} K$ be distinct, then there exists a hypergraph Z_e^X and a finite index full relatively quasiconvex $K_W \leq \operatorname{Stab}_G(Z_e^X)$ so that p and q are in K_W–distinct components of $\partial_\mathcal{D} K \smallsetminus \Lambda K_W$.*

Proof. By Proposition 6.10 there exists a hypergraph that separates p and q. The action of G on $X_0^{(1)}$ has trivial edge stabilizers, the action is cocompact and every maximal parabolic fixes a vertex. Therefore, by Theorem 5.15, there exists a finite index subgroup K_W of $\mathrm{Stab}_G(Z_e^X)$ so that p and q lie in K_W-distinct components of $\partial_{\mathcal{D}} K \smallsetminus \Lambda K_W$. Full relative quasiconvexity is preserved by taking finite index subgroups, so full relative quasiconvexity of K_W is implied by Hypotheses 6.2. $\square$

7. WHY X_{bal} HAS SUITABLE WALLS

We easily obtain Hypotheses 6.2 Items (1) and (2). Each hypergraph Z_e^X is an embedded tree whose intersection with $X_0^{(1)}$ is quasi-convex in $X_0^{(1)}$ by [MS17, Theorem 3.10] showing (1).

Moreover, [MS17, Proposition 3.14] shows that each Z_e^X has exactly two complementary components proving (2).

7.1. **Fullness.** In this subsection, we establish Hypothesis 6.2(3) and some more general results that will help us in the blowup case below.

We first establish that hyperplane stabilizers for relatively geometric actions are full. Proposition 7.1 will help motivate our technique for proving Proposition 7.2 below and will be used later.

Proposition 7.1. *Let $(K, \mathcal{D})$ act relatively geometrically on a CAT(0) cube complex $\widetilde{X}$. If $L \leq K$ is a hyperplane stabilizer, then L is full.*

Proof. Let W be the hyperplane of $\widetilde{X}$ stabilized by L. Suppose that there exist $k \in K$ and $D \in \mathcal{D}$ so that $L \cap D^k$ is infinite. K acts relatively geometrically, so every maximal parabolic stabilizes a vertex and every cell stabilizer is finite index in a maximal parabolic. Then there exists a finite index subgroup $D_v \leq D^k$ that stabilizes a vertex so that $D_v \cap L$ is infinite.

Claim: an infinite subgroup of D^k fixes an edge e of $\widetilde{X}$ dual to W. By convexity of W, the nearest point projection of v to W with respect to the CAT(0) metric is well-defined and is a unique point. Let $p_0 \in W$ denote this point. Since $p_0 \in W$, p_0 lies in the interior of some cube M of $\widetilde{X}$ that has an edge dual to W. Since K acts on $\widetilde{X}$ by isometries, the uniqueness of p_0 implies that $(D_v \cap L)$ fixes p_0. Since $\mathrm{Stab}_K(p_0)$ fixes the cube M setwise, a finite index subgroup $D_1 \leq \mathrm{Stab}_K(p_0)$ fixes M pointwise. Then $D_M = D_1 \cap D_v \cap L$ is an infinite subgroup of $D_v \cap L \leq D^k$ that fixes M and fixes an edge e dual to W. This completes the proof of the claim.

Claim: the stabilizer of e is finite index in D^k. Since $\mathrm{Stab}_K(e)$ is infinite, $\mathrm{Stab}_K(e)$ is finite index in a maximal parabolic. By construction, $\mathrm{Stab}_K(e)$ has infinite intersection with D^k. Thus $\mathrm{Stab}_K(e)$ is finite index in D^k because distinct maximal parabolics have finite intersections. This completes the proof of the second claim.

Proposition 7.1 now follows because the K–stabilizer of e is contained in L. $\square$
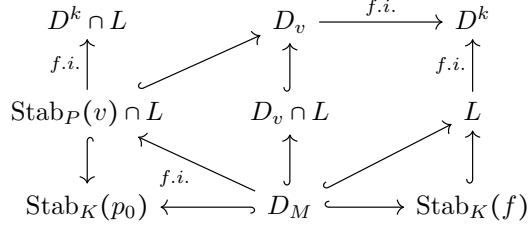
The following Proposition now implies Hypotheses 6.2(3):

FIGURE 5. Commutative diagram depicting the various subgroups considered in the proof of Proposition 7.1.

Proposition 7.2. *Let Z be a hypergraph in X_{bal} with stabilizer H and let v be a vertex of X_{bal} with stabilizer G_v. If v is not contained in Z, then $G_v \cap H$ is finite.*

Recall that as a hypergraph, Z can be either an X_{bal}–hyperstructure or a projected hypergraph, the image of an $\mathcal{E}G_{bal}$–hyperstructure under the natural projection $\mathcal{E}G_{bal} \to X_{bal}$. We first prove an auxiliary Lemma:

Lemma 7.3. *Let Y be the hypercarrier of Z in X_{bal}, let v be a vertex of X_{bal}, and let $D = \min\{d_{X_{bal}^{(1)}}(z,v) : z \in Y^{(0)}\}$. Then $S := \{z \in Y^{(0)} : d_{X_{bal}^{(1)}}(z,v) = D\}$ is finite.*

Proof. We proceed by induction on D. When $D = 0$, the statement is obvious.

Suppose toward a contradiction that S is infinite. Choose a distinguished vertex $x \in S$ and γ a geodesic path in $X_{bal}^{(1)}$ from v to x. Let $S \smallsetminus \{x\} = \{x_1, x_2, \dots\}$ be an enumeration[1] of vertices distinct from x. Choose geodesic paths γ_i in $X_{bal}^{(1)}$ from v to x_i with the following property: if v_1 and v_2 are vertices of γ_i that are also vertices of γ, then γ_i and γ follow the same path between v_1 and v_2. If all but finitely many γ_i leave v by following the same edge, then there exists a vertex v' so that $d_{X_{bal}^{(1)}}(v', Y^{(0)}) = D - 1$ and there are infinitely many vertices in Y that realize the minimal distance $D - 1$. By induction, we obtain a contradiction.

Therefore, by possibly passing to an infinite subset of the γ_i, we may assume the γ_i only intersect at v. Note that $\gamma_i \cap Y = \{x_i\}$ and $\gamma \cap Y = \{x\}$ because $x, x_i \in S$. Then there exists a path $\sigma_i \subseteq Y \cap X_{bal}^{(1)}$ from x to x_i of length at most $2D$ because $Y^{(1)}$ is convex. Then the paths $\rho_i = \gamma_i \cup \sigma \cup \gamma$ are circuits based at v in $X_{bal}^{(1)}$ of length at most $4D$ that include the edges of γ. There are infinitely many ρ_i which violates the fineness of $X_{bal}^{(1)}$. $\square$

Proof of Proposition 7.2. Recall that Y denotes the hypercarrier of Z and $H = \operatorname{Stab}_G(Z)$. Let $H_v := G_v \cap H$. First, assume that $v \notin Y^{(0)}$. Then there exists a vertex $x \neq v$ so that $x \in Y^{(0)}$ minimizes the (1–skeleton) distance from Y to v by Lemma 7.3.

We claim that H_v is commensurable to a subgroup of $G_v \cap G_x$. Indeed, by Lemma 7.3 the G_v orbit of x is finite. Hence, H_v has a finite index subgroup which stabilizes x. Stabilizers of distinct

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[1]Recall $X_{bal}^{(1)}$ is a subdivided quotient of a tree with countably many vertices and therefore has countably many vertices.

vertices in X_{bal} have trivial intersection because the local group at each edge is trivial. Hence, H_v is finite.

It remains to show the case where $v \in Y^{(0)}$. Since Z does not contain v, there exists a polygon $U \subseteq Y$ so that $v \in U$ and the interior of U intersects Z.

If U is the unique polygon of Y containing v then H_v stabilizes U. Moreover, H_v acts on U as a subgroup of a finite dihedral group, so a finite index subgroup fixes U pointwise. In particular, this finite index subgroup fixes an edge, so H_v is finite because edge stabilizers are trivial.

Hence, suppose there are multiple polygons of Y that contain v. Let

$$\mathcal{U} = \{U : \text{U is a polygon of } Y \text{ that contains } v\}.$$

Claim: if $U \in \mathcal{U}$, then ∂U has at most two edges dual to Z. The abstract carrier of Z embeds by Proposition 4.8. The abstract carrier has one copy of U for every pair of diametrically opposed edges dual to Z. Since the natural map from the abstract carrier to Y is injective on the interior of polygons, there can only be one polygon of the abstract carrier whose interior maps to the interior of U.

Claim: there is a path σ from v to Z so that if $U \in \mathcal{U}$, then ∂U contains σ. Let $U_1, U_2 \in \mathcal{U}$ with $U_1 \neq U_2$. Since the abstract carrier of Z embeds in X_{bal}, the copies of U_1, U_2 in the abstract carrier must intersect because $v \in U_1 \cap U_2$. The construction of the abstract carrier (see Definition 4.2) requires that if the copies of U_1, U_2 intersect, then U_1, U_2 must share an edge e dual to Z and a path σ without backtracking from e to v lies in $\partial U_1 \cap \partial U_2$.

Since U_1 may have two edges dual to Z, there at most two possibilities for σ. However, σ is a piece, so $|\sigma| \leq \frac{1}{6}|\partial U_1|$ by the $C'(1/6)$ condition. Therefore, there is only one choice for σ.

Claim: if $h \in H_v$, then $h \cdot \sigma = \sigma$. Recall that the action of G on X_{bal} takes polygons to polygons. Observe that $H_v = G_v \cap H$ fixes Z setwise and fixes v. Therefore, if $U \in \mathcal{U}$, then $H_v \cdot U \in \mathcal{U}$. Since H_v fixes Z and v, $h \cdot \sigma$ is a path from v to Z of length at most $\frac{1}{6}|\partial U|$. Since U has two edges dual to Z, $h \cdot \sigma$ must be σ. This completes the proof of the claim.

Finally, since any $h \in H_v$ must fix σ pointwise and $v \notin Z$, H_v fixes an edge of X_{bal} pointwise. However, edge stabilizers of X_{bal} are trivial, so we conclude H_v is trivial. $\qquad \square$

Recall that a fiber complex is the preimage of a vertex of X_{bal} under the projection.

Lemma 7.4. *Let $H = \mathrm{Stab}_G(W_e^{\mathcal{E}G})$, let G_v be the stabilizer of a vertex v of X and let EG_v be the associated fiber complex. If $W_e^{\mathcal{E}G}$ intersects EG_v, then:*

(1) $W_e^{\mathcal{E}G} \cap EG_v$ *is a hyperplane.*
(2) $H \cap G_v$ *is the stabilizer of a hyperplane of EG_v.*

Proof. For the first claim, one might worry that $W_e^{\mathcal{E}G}$ might intersect $\mathcal{E}G_v$ in more than one hyperplane, but then the projection $W_e^{\mathcal{E}G} \to Z_e^{\mathcal{E}G}$ would contain a loop. By [MS17, Lemma 3.32], $Z_e^{\mathcal{E}G}$ is an embedded tree in X_{bal}. This proves the first claim.

Let $W_v = W_e^{\mathcal{E}G} \cap EG_v$ so that W_v is a hyperplane of EG_v. If $h \in H \cap G_v$, then $h \cdot W_v$ is a hyperplane of EG_v, so $h \cdot W_v = W_v$. Therefore $H \cap G_v \subseteq \mathrm{Stab}_{G_v}(W_v)$. Similarly, if $h' \in \mathrm{Stab}_{G_v}(W_v)$, then the intersection of $h' \cdot W_e^{\mathcal{E}G}$ and EG_v is W_v. Then $h' \cdot W_e^{\mathcal{E}G} = W_e^{\mathcal{E}G}$ because $h' \cdot W_e^{\mathcal{E}G}$ is an $\mathcal{E}G$–hyperstructure and $W_e^{\mathcal{E}G}$ is the unique hyperstructure that intersects EG_v in the hyperplane W_v. Therefore, $\mathrm{Stab}_{G_v}(W_v) \subseteq G_v \cap H$. $\qquad\square$

Proposition 7.5. *Let $H = \mathrm{Stab}_G(W)$ where $W = Z_e^X$ or $W = W_e^{\mathcal{E}G}$. Then H is full in $(G, \mathcal{P})$.*

Proof. First suppose that $W = Z_e^X$. Since W contains no vertices, $|G_v \cap H| < \infty$ for all vertices v of X_{bal} by Proposition 7.2. Every peripheral subgroup is contained in a fiber group, so H has finite intersection with any peripheral subgroup.

Now suppose $W = W_e^{\mathcal{E}G}$.

If W intersects EG_v, then Lemma 7.4 implies that the $G_v \cap H$ stabilizes a hyperplane of EG_v. Hyperplane stabilizers in EG_v are full (Proposition 7.1), so for any peripheral subgroup $P \in \mathcal{P}_v$ contained in G_v, $H \cap P$ is either finite or finite index in P.

Observe that H stabilizes $Z_e^{\mathcal{E}G}$, a (projected) hypergraph in X_{bal}. Recall $Z_e^{\mathcal{E}G}$ intersects a vertex v of X_{bal} if and only if W intersects EG_v. Therefore, if v is a vertex of X_{bal} so that W does not intersect EG_v then by Proposition 7.2:

$$|G_v \cap H| \le |G_v \cap \mathrm{Stab}_G(Z_e^{\mathcal{E}G})| < \infty$$

Since every peripheral subgroup is contained in a fiber group, it follows that H is full in $(G, \mathcal{P})$. $\quad\square$

7.1.1. *Infinite geodesics in X_{bal} have many dual hypergraphs that cross exactly once.* The main result of this subsection is:

Proposition 7.6. *Let $\gamma \subseteq X_{bal}^{(1)}$ be a combinatorial geodesic in $X_{bal}^{(1)}$. There exists an edge e of γ_0 so that the hypergraph Z_e^X intersects γ exactly once. Further, if γ is infinite, there exists $N \in \mathbb{N}$ so that every collection of at least N consecutive edges contains an edge e so that Z_e^X crosses γ exactly once.*

Before proving Proposition 7.6, we introduce some facts about small cancellation, and prove Lemmas 7.7 and 7.8 which will be used in the proof of Proposition 7.6.

Lemma 7.7. *Let γ be a geodesic in $X^{(1)}$ containing an edge e and let Z_e^X be the dual hypergraph. Suppose γ is dual to Z_e^X at another edge e' of γ. Let γ_e^X be the unique path in Z_e^X between e and e' that does not backtrack. Then there is a sequence of polygons $U_1, U_2, \ldots, U_m$ of X_{bal} so that:*

(1) *U_i has two edges dual to γ_e^X (equivalently γ_e^X intersects the interior of U_i),*

(2) *the subpath of γ from e to e' including these two edges is contained in $\bigcup_{i=1}^m \partial U_i$, and*

(3) *$\gamma \cap \partial U_i \neq \varnothing$ for all $1 \le i \le m$.*

Proof. Following γ_e^X from e to e', we label the sequence of polygons that γ_e^X passes through the interiors of as $U_1, U_2, \ldots, U_n$ (so that each U_i has two edges dual to γ). The collection $U_1, U_2, \ldots, U_n$

are a **gallery** in the sense of [MS17, Definition 3.7] (where the **doors** are the edges dual to γ_e^X). The union of the U_i in X_{bal} is the image of the associated hypercarrier in the sense of [MS17, Definition 3.8]. Then [MS17, Proposition A.16] implies that γ lies in $\bigcup_i \partial U_i$.

Let $e = e_0, e_1, \ldots, e_k = e'$ be the edges of γ in the interval between and including e, e'.

Claim: if U, V are distinct polygons of the carrier of Z_e^X so that $e_i \subseteq \partial U$ and $e_{i+1} \subseteq \partial V$, then there is an edge in $U \cap V$ that is dual to Z_e^X. In the abstract carrier of Z_e^X, there are polygons $\overline{U}, \overline{V}$ containing edges $\overline{e}_i$ and $\overline{e}_{i+1}$ respectively so that the natural map carries $\overline{U}, \overline{V}, \overline{e}_i, \overline{e}_{i+1}$ to U, V, e_i, e_{i+1} respectively. Since the abstract carrier embeds (recall Proposition 4.8), $\overline{e}_i$ and $\overline{e}_{i+1}$ must intersect in a vertex because their images e_i, e_{i+1} share a vertex. By construction of the abstract carrier, any two polygons that have nontrivial intersection in the abstract carrier of Z_e^X share an edge dual to Z_e^X. This proves the claim.

We may assume that $U_1, U_2, \ldots, U_n$ are listed in the sequence they appear by traversing γ_e^X from e to e' (recalling that γ_e^X is a subpath of a tree that does not backtrack). Next we claim that if $U_i \cap U_j \neq \varnothing$, then $j \in \{i-1, i, i+1\}$. For $i < n$ U_i, U_{i+1} intersect in an edge dual to Z_e^X and if $j \neq i, i+1$, U_j cannot contain the same dual edge or γ_e^X would not be a subpath of an embedded tree. Similarly, for $j \neq i, i-1$ and $i > 1$, U_j cannot contain the edge of γ_e^X dual to edges of U_i, U_{i-1}. Then the preceding claim implies that among $U_1, \ldots, U_n$, U_i may only intersect itself and $U_{i\pm 1}$.

Thus a path from U_1 to U_n must intersect every U_i for $1 \leq i \leq n-1$. $\qquad\square$

Lemma 7.8. *Let γ be a geodesic in $X^{(1)}$. Let γ_e^X be an embedded geodesic path in Z_e^X between edges e and another distinct edge e' on γ and let U be a polygon containing e where γ_e^X is dual to e and a second edge $\widehat{e}$ of ∂U. Then*

(1) *the edge $\widehat{e}$ satisfies $d_{X_{bal}^{(1)}}(\widehat{e}, \gamma \cap \partial U) < \frac{1}{6}|\partial U| - 1$, and*
(2) *$\frac{1}{3}|\partial U| < |\gamma \cap \partial U| \leq \frac{1}{2}|\partial U|$.*

Proof. By Lemma 7.7 the path γ_e^X lies entirely in polygons whose boundary intersects γ. Therefore, γ_e^X must exit U at $\widehat{e}$ through a piece of ∂U that intersects γ, so the first claim follows immediately by the $C'(\frac{1}{6})$ condition.

Since γ is a geodesic, we have the upper bound $|\gamma \cap \partial U| \leq \frac{1}{2}|\partial U|$. For the lower bound in Item 2, e lies on γ and $e, \widehat{e}$ are diametrically opposed, so $d_{X_{bal}^{(1)}}(e, e') \geq \frac{1}{2}|\partial U| - 1$. With the first claim,

$$|\gamma \cap \partial U| \geq |e| + d_{X_{bal}^{(1)}}(e, \widehat{e}) - d_{X_{bal}^{(1)}}(\widehat{e}, \gamma \cap \partial U) > 1 + \frac{1}{2}|\partial U| - \left(\frac{1}{6}|\partial U| - 1\right) > \frac{1}{3}|\partial U|. \qquad\square$$

Proof of Proposition 7.6. Let f be an edge of γ. If Z_f^X intersects γ exactly once, we are done. Thus, suppose $f' \subset \gamma$ is an edge such that $f \neq f'$ and Z_f^X intersects f' nontrivially. Since f and f' are distinct, the definition of a hypergraph implies there exists a polygon U in the carrier of Z_f^X with $f \subset \partial U$. We will use $\gamma \cap \partial U$ to detect a hypergraph crossing γ exactly once.

Order edges of γ and label them consecutively by e_i for $i \in \mathbb{Z}$ such that $\gamma \cap \partial U = e_1 \cup \cdots \cup e_k$. By Lemma 7.8, we have that $\frac{1}{3}|\partial U| \le k \le \frac{1}{2}|\partial U|$. Hence, we may choose i between 1 and k such that

$$\frac{1}{6}|\partial U| - 1 < \min\{d_{X_{bal}^{(1)}}(e_i, e_1), d_{X_{bal}^{(1)}}(e_i, e_k).\} \tag{$\star$}$$

In particular, every piece of ∂U containing e_1 or e_k is disjoint from e_i by the $C'(\frac{1}{6})$ condition. Let $e \coloneqq e_i$ for the above choice of i and let Z_e^X be the hypergraph through e. We will see that Z_e^X only crosses γ at a single point (the midpoint of e).

Assume for contradiction that there exists an edge $e' \subset \gamma$ with $e' \ne e$ and let $\gamma_e^X \subseteq Z_e^X$ be a geodesic segment joining the midpoints of e and e'. Let $\check{U}$ denote the polygon with $e \subseteq \partial\check{U}$ so that the interior of $\check{U}$ has nontrivial intersection with γ_e^X. We now prove $\check{U} \ne U$. Indeed, let $\widehat{e} \subset \partial U$ be the edge opposite to e in ∂U. Clearly, $\widehat{e} \cap \gamma = \varnothing$, and the inequality in Equation ($\star$) immediately implies the same inequality with e replaced by $\widehat{e}$. Hence, γ_e^X cannot cross $\widehat{e}$ as this would contradict Lemma 7.8(1).

We specially label one more edge of γ to complete this proof. In the labelling of γ let $e_\ell \coloneqq e'$. Either $\ell < 1$ or $\ell > k$ by convexity of hypercarriers. If $\ell < 1$ fix $j = 0$, otherwise, fix $j = k + 1$. By choice of the labelling, e_j is the edge closest to e on the subinterval of γ between e and e' whose interior is disjoint from $\gamma \cap \partial U$. There exists a polygon $W \ne U$ that is adjacent to e_j and whose interior has nontrivial intersection with γ_e^X by Lemma 7.7.

Note that $\check{U} \ne W$ for otherwise $\partial\check{U}$ would contain the entire subinterval of γ joining e and e_j. Such a subinterval would yield a piece between $\partial\check{U}$ and ∂U containing e and one of e_1 or e_k, which is impossible by how we chose of e.

Martin and Steenbock showed that hypergraphs in X are embedded trees [MS17, Theorem 3.10]. It follows that any polygon whose interior intersects Z_e^X disconnects the carrier Y_e^X. In particular, U and W must lie in distinct components of $Y_e^X \smallsetminus \check{U}$ by construction of γ_e^X. However, U and W have nontrivial intersection containing an endpoint of e_j a contradiction. Thus, Z_e^X crosses γ exactly once.

Since G acts cocompactly on X_{bal}, there exists some $N \in \mathbb{N}$ so that any polygon U of X_{bal} has $|\partial U| \le N$. Based on the above argument, every length $2N$ subsegment of γ must contain some edge e so that Z_e^X intersects γ exactly once. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

8. Relatively cubulating blowups of polygonal complexes

In this section we provide a general framework for establishing relatively geometric actions for refined peripheral structures of a given relatively hyperbolic groups that acts on a polygonal complex whose 1–skeleton is a fine hyperbolic graph.

8.1. **Relatively Geometric Blowups.**

Definition 8.1. *Let $(K, \mathcal{D})$ be a relatively hyperbolic pair acting on a polygonal complex X_0 satisfying Hypotheses 6.2. A **relatively geometric blowup** of $(X_0, K, \mathcal{D})$ is a cell complex $\mathcal{E}X_0$ with a K–action and a K–equivariant surjective combinatorial projection $\pi : \mathcal{E}X_0 \to X_0$. so that:*

(1) *$\mathcal{E}X_0$ is simply connected and the cells are either n–cubes or even-sided polygons.*

(2) *K acts cocompactly by isometries on $\mathcal{E}X_0$.*

(3) *For any $x \in X_0$, the π–fiber is a (connected) CAT(0) cube complex, and if $\pi^{-1}(x)$ contains more than one point then x is a vertex.*

(4) *every cell not in the π–fiber of a vertex is an edge or an even sided polygon.*

(5) *Each $D \in \mathcal{D}$ admits a relatively hyperbolic structure $(D, \mathcal{R}_D)$ so that if v is a vertex stabilized by D then $(D, \mathcal{R}_D)$ acts relatively geometrically on $\pi^{-1}(v)$.*

*If v is a vertex, we call $\pi^{-1}(v)$ a **fiber complex**.*

The blowup in Definition 8.1 is a modification of that introduced by Martin and Steenbock [MS17] who replace condition (5) with D acting geometrically on a CAT(0) cube complex. Observe that the equivariance of the projection ensures that for $g \in K$, D^g acts relatively geometrically on the fiber complex over the vertex stabilized by D^g with respect to $\mathcal{R}_D^g = \{R^g : R \in \mathcal{R}_D\}$.

Remark 8.2. *Note that X_0 with the identity map is a relatively geometric blowup of itself where the peripheral structure on a stabilizer K_v of a vertex v is $(K_v, \{K_v\})$, which acts relatively geometrically on v by the trivial action.*

Notation 8.3. *Given $D \in \mathcal{D}$ we write ED for the fiber complex over the vertex stabilized by D.*

Suppose that $(\mathcal{E}X_0, \pi)$ is a relatively geometric blowup of $(X_0, K, \mathcal{D})$. Then each $D \in \mathcal{D}$ stabilizes a vertex of X_0 and has a peripheral structure $\mathcal{R}_D$. If we let $\mathcal{R} = \bigcup_{D \in \mathcal{D}} \mathcal{R}_D$, then $(K, \mathcal{R})$ is a relatively hyperbolic pair. We call $(K, \mathcal{R})$ the **refined peripheral structure for the blowup**.

The following is a straightforward consequence of the fact that the action of K on X_0 witnesses the relative hyperbolicity of $(K, \mathcal{R})$ and [Yan14, Theorem 1.1]:

Proposition 8.4. *If v is a vertex of X_0, then $\operatorname{Stab}_K(v)$ is relatively quasiconvex in $(K, \mathcal{R})$.*

Thus if $D \in \mathcal{D}$, we obtain an inclusion $\partial_{\mathcal{R}_D} D \hookrightarrow \partial_{\mathcal{R}} K$ whose image is the limit set of D.

We recall a useful result that will be used for fullness:

Proposition 8.5 ([EGN24, Proposition 6.1]). *Let W be a hyperplane of ED, the fiber complex stabilized by D and let R_{ED} be a maximal parabolic subgroup in $(D, \mathcal{R}_D)$. W is dual to an edge whose stabilizer is commensurable to R_{ED} if and only if R_{ED} is commensurable to a subgroup of $\operatorname{Stab}_D(W)$.*

8.2. Separating points in the Bowditch Boundary of a fiber complex stabilizer.

Recall that our goal is to use Theorem 2.7 to produce a relatively geometric action by finding full relatively

quasiconvex subgroups that separate any pair of points in the Bowditch Boundary. The next result shows that Theorem 6.12 is enough provided that we can separate pairs of points in the boundary of a single fiber complex stabilizer:

Corollary 8.6. *Let $p, q \in \partial_{\mathcal{R}} K$ be distinct. If p, q are not points in the Bowditch Boundary of a single fiber complex stabilizer, then there exists a full relatively quasiconvex subgroup $H_{p,q}$ in $(K, \mathcal{R})$ so that p, q are in $H_{p,q}$–distinct components of $\partial_{\mathcal{R}} G \smallsetminus \Lambda H_{p,q}$.*

Proof. Recall from Proposition 5.1 that there exists a continuous map $\phi : \partial_{\mathcal{R}} G \to \partial_{\mathcal{D}} G$ and $\phi(p) \neq \phi(q)$ if they are not in the boundary of a single fiber group. Theorem 6.12 and Proposition 5.2 now imply the desired result. $\square$

[EGN24, Theorem 1.4] implies that any two distinct points in the Bowditch Boundary of a group that acts relatively geometrically can be separated into distinct complementary components of the limit set of a hyperplane stabilizer and a finite index subgroup of the hyperplane stabilizer does not identify these components up to group action. Every hyperplane in a fiber complex is the intersection of that fiber complex with an $\mathcal{E}X_0$–hyperstructure $W_e^{\mathcal{E}X_0}$. We aim to use this fact by showing that we can extend the stabilizers of these hyperplanes to codimension-one subgroups of K. We recall two useful results that were used in the process of proving [EGN24, Theorem 1.4]:

Lemma 8.7 ([EGN24, Lemma 6.7])**.** *Let $(L, \mathcal{F})$ act relatively geometrically on a CAT(0) cube complex $\widetilde{C}$. Let $\gamma : (-\infty, \infty) \to \widetilde{C}^{(1)}$ be a connected bi-infinite combinatorial (λ, ε)–quasi-geodesic. Given $M > 1$, there exist a hyperplane W of $\widetilde{C}$ and $t_M > 0$ so that for all t with $|t| > t_M$:*

  (1) *$\gamma(\pm t)$ are separated by W,*
  (2) *γ crosses W an odd number of times, and*
  (3) *$d(\gamma(t), W) > M$.*

Also, $\gamma(t) \in W$ implies $|t| \le t_M$.

Lemma 8.8 ([EGN24, Lemma 6.8])**.** *Let $(L, \mathcal{F})$ act relatively geometrically on a CAT(0) cube complex $\widetilde{C}$. Let $\gamma : [0, \infty) \to \widetilde{C}$ be an infinite combinatorial (λ, ε)–quasi-geodesic ray where $x = \gamma(0)$ is a vertex. Given $M > 1$, there exists a hyperplane W of $\widetilde{C}$ and $t_M > 0$ so that for all t with $t > t_M$:*

  (1) *W separates $\gamma(0)$ and $\gamma(t)$,*
  (2) *if $\mathrm{Stab}_L(x)$ is infinite, W is not dual to any edge whose stabilizer is commensurable to $\mathrm{Stab}_L(x)$, and*
  (3) *$d(\gamma(t), W) > M$.*

In particular, if $\gamma(t) \in W$, then $0 < t \le t_M$.

Recall that the vertex groups D^g where $D \in \mathcal{D}$ and $g \in K$ act relatively geometrically on their associated fiber complex ED^g. This gives the 1–skeleton, $(ED^g)^{(1)}$, the structure of a generalized fine hyperbolic graph with respect to this action. Generalized fine graphs are easier to work with

for separating boundary points as was shown in [EGN24]. In particular, the complete electrification map described in Proposition 5.5 is an equivariant quasi-isometry to a fine hyperbolic graph.

Proposition 8.9. $\mathcal{E}X_0^{(1)}$ *is a generalized fine hyperbolic graph with respect to the action of* $(K, \mathcal{R})$.

Proof. By [CC07, Theorem 5.1] (a general version of Proposition 3.7), $\mathcal{E}X_0$ is equivariantly quasi-isometric to the coned-off Cayley graph for $(K, \mathcal{R})$. Since the actions of the fiber groups on the fiber complexes are relatively geometric, every maximal parabolic stabilizes a vertex of $\mathcal{E}X_0$. Likewise, if Σ_{R^g} is the sub-graph of $\mathcal{E}X_0^{(1)}$ whose cells have stabilizer commensurable to R^g for some $R \in \mathcal{R}$ and $g \in K$, then Σ_{R^g} lies in a fiber complex and is connected and compact by [EG22, Proposition 3.5]. If Γ is the complete electrification of $\mathcal{E}X_0$ with respect to Σ_{R^g}, then Γ has trivial edge stabilizers, and the maximal parabolic subgroups of $(K, \mathcal{R})$ each stabilize exactly one vertex of Γ. Therefore, by Proposition 3.7, Γ is also quasi-isometric to the coned-off Cayley graph for $(K, \mathcal{R})$ and is hyperbolic. Hence, by [Bow12, Lemma 4.5], Γ is a fine hyperbolic graph. Therefore by [EGN24, Proposition 3.2], $\mathcal{E}X_0^{(1)}$ is generalized fine with respect to the action of $(K, \mathcal{R})$. $\square$
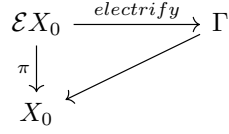

FIGURE 6. Diagram relating the projection map π with the electrification in the proof of Proposition 8.9.

We fix the following notation for the remainder of this section. Let $p, q \in \partial_{\mathcal{R}_D}(D)$ be distinct. Recall ED is the fiber complex over a vertex stabilized by D. Then by [EGN24, Theorem 1.4], there exists a hyperplane W_D in ED so that $\Lambda \operatorname{Stab}_D(W_D)$ separates p and q in $\partial_{\mathcal{P}_D} D$. Let e be an edge dual to W_D. Then $W_e^{\mathcal{E}X_0}$ contains W_D in $\mathcal{E}X_0$.

Let $\beta : \mathcal{E}X_0^{(1)} \to \Gamma$ be the equivariant quasi-isometry that collapses the Σ_{D^g} from Proposition 8.9 where Γ is a fine hyperbolic graph witnessing the relative hyperbolicity of $(K, \mathcal{R})$ in the sense of Definition 2.1. Let L be the hyperset associated to $W_e^{\mathcal{E}X_0}$ in $\mathcal{E}X_0$ and let J be the associated hypercarrier.

We now introduce a few additional conditions on the blowup that will be helpful to control projections of the hyperstructures $W_e^{\mathcal{E}X_0}$:

Definition 8.10. *Let* $(\mathcal{E}X_0, \pi)$ *be a relatively geometric blowup of* $(X_0, K, \mathcal{D})$ *as in Definition 8.1.*

  (6) $(\mathcal{E}X_0, \pi)$ *has the **projected wall tree property** if* π *projects every* $\mathcal{E}X_0$–*hyperstructure to an (embedded) tree in* X_0 *whose intersection with* $X_0^{(1)}$ *is quasiconvex.*
  (7) $(\mathcal{E}X_0, \pi)$ *has the **projected wall fullness property** if the following holds: whenever* $W_e^{\mathcal{E}X_0}$ *is a* $\mathcal{E}X_0$–*hyperstructure and* v *is a vertex of* X_0 *so that* $\operatorname{Stab}_K(W_e^{\mathcal{E}X_0})$ *is infinite,* $\operatorname{Stab}_K(W_e^{\mathcal{E}X_0})$ *has infinite intersection with* $\operatorname{Stab}_K(v)$ *if and only if* $\pi(W_e^{\mathcal{E}X_0})$ *intersects* v.

(8) $(\mathcal{E}X_0, \pi)$ has the **two-sided wall projection property** if for every $W_e^{\mathcal{E}X_0}$, π maps the complementary components of $W_e^{\mathcal{E}X_0}$ in $\mathcal{E}X_0$ to distinct components of $\pi(W_e^{\mathcal{E}X_0})$.

Note that Item 8 verifies that the projection of the wall space structure on the blow-up induces a wallspace structure on base polygonal complex.

Proposition 8.11. *If $(\mathcal{E}X_0, \pi)$ has the projected wall tree property, then $\mathrm{Stab}_K(W_e^{\mathcal{E}X_0})$ is relatively quasiconvex in $(K, \mathcal{D})$.*

Proof. $\mathrm{Stab}_G(W_e^{\mathcal{E}X_0})$ acts cocompactly on the quasiconvex tree $\pi(W_e^{\mathcal{E}X_0})$ by Lemma 4.4. Therefore, Theorem 2.2 implies that $\mathrm{Stab}_G(W_e^{\mathcal{E}X_0})$ is relatively quasiconvex in $(K, \mathcal{D})$. $\qquad\square$

Proposition 8.12. *Suppose the blow-up $(\mathcal{E}X_0, \pi)$ has the projected wall tree and fullness properties. Let $H = \mathrm{Stab}_K(W_e^{\mathcal{E}X_0})$. Then:*

(1) *If ED^g is a fiber complex and $W_e^{\mathcal{E}X_0}$ intersects ED^g, then $W_e^{\mathcal{E}X_0} \cap ED^g$ is a hyperplane and $H \cap \mathrm{Stab}_K(v)$ is the stabilizer of a hyperplane of ED^g.*
(2) *H is full relatively quasiconvex in $(K, \mathcal{R})$.*

Proof. For the first claim, if $W_e^{\mathcal{E}X_0}$ intersects ED^g in more than one hyperplane, then the projection $\pi(W_e^{\mathcal{E}X_0})$ will contain a loop. This contradicts the fact that $\pi(W_e^{\mathcal{E}X_0})$ is a tree.

Let $W_{D^g} = W_e^{\mathcal{E}X_0} \cap ED^g$ so that W_{D^g} is a hyperplane of ED^g. If $h \in H \cap D^g$ then $h \cdot W_v$ is a hyperplane of ED^g. Therefore, $h \cdot W_{D^g} = W_{D^g}$ because $ED^g \cap W_e^{\mathcal{E}X_0} = W_{D^g}$. Hence $H \cap D^g \leq \mathrm{Stab}_{D^g}(W_{D^g})$. Similarly, if $h' \in \mathrm{Stab}_{D^g}(W_{D^g})$, then the intersection of $h' \cdot W_e^{\mathcal{E}X_0}$ with ED^g is W_{D^g}. Then $h' \cdot W_e^{\mathcal{E}X_0} = W_e^{\mathcal{E}X_0}$ because $W_e^{\mathcal{E}X_0}$ is the unique $\mathcal{E}X_0 - -$hyperstructure that intersects ED^g in W_{D^g}. Then $\mathrm{Stab}_{D^g}(W_{D^g}) \leq D^g \cap H$.

Relative quasiconvexity of H in $(K, \mathcal{D})$ follows from Proposition 8.11. By [Yan14, Theorem 1.3], H is relatively quasiconvex in $(K, \mathcal{R})$ exactly when $H \cap Q^g$ is relatively quasiconvex in $(K, \mathcal{R})$ for all $g \in G$ and $Q \in \mathcal{D}$. Since Q^g stabilizes a fiber complex, $H \cap Q^g$ is a hyperplane stabilizer which is relatively quasiconvex in $(K, \mathcal{R})$ by Proposition 8.16.

If $W_e^{\mathcal{E}X_0}$ intersects ED^g, then $D^g \cap H$ stabilizes a hyperplane of ED^g. Hyperplane stabilizers of relatively geometric actions are full by Proposition 7.1 so for any parabolic (with respect to $(K, \mathcal{R})$) subgroup R of D^g, $H \cap R$ is either finite or finite index in R.

By the projected wall fullness property, H intersects the stabilizer of a fiber complex associated to a vertex v of X_0 exactly when $\pi(W_e^{\mathcal{E}X_0})$ passes through v. Therefore, if $W_e^{\mathcal{E}X_0}$ does not intersect the fiber complex over v:

$$|\mathrm{Stab}_K(v) \cap H| \leq |\mathrm{Stab}_K(v) \cap \mathrm{Stab}_K(p(W_e^{\mathcal{E}X_0}))| < \infty.$$

Since every maximal parabolic is contained the stabilizer of a fiber complex, it follows that H is full in $(K, \mathcal{R})$. $\qquad\square$

Since $\mathrm{Stab}_G(W_e^{\mathcal{E}X_0})$ is relatively quasiconvex in $(K, \mathcal{R})$ by Proposition 8.11, $L = W_e^{\mathcal{E}X_0} \cap \mathcal{E}X_0^{(1)}$ is quasiconvex in $\mathcal{E}X_0^{(1)}$, and hence the associated hypercarrier J is quasiconvex. The following property follows immediately from the two-sided wall projection property and the fact that π is continuous:

Proposition 8.13. *If $(\mathcal{E}X_0, \pi)$ has the two-sided wall projection property, each W_e^{EG} separates $\mathcal{E}X_0$ into two complementary components which naturally divides J into J^+ and J^- so that $J^+ \cap J^- = L$.*

Further, every path between vertices of components of $\mathcal{E}G_{bal}^{(1)} \smallsetminus L$ must traverse an edge dual to $W_e^{\mathcal{E}G}$.

By Proposition 5.9, $\beta(L)$ is a hyperset in the fine hyperbolic graph Γ and $\beta(J)$ is its hypercarrier. Moreover, because we are working with the refined structure $(K, \mathcal{R})$ each of the stabilized sub-graphs Σ_{R^g} for $R \in \mathcal{R}$ and $g \in K$ is contained in a fiber complex.

Proposition 8.14. *If $(\mathcal{E}X_0, \pi)$ has the two-sided wall projection, projected wall tree and projected wall fullness property, then the hypercarrier $\beta(J)$ satisfies the two-sided carrier property (Recall Definition 5.11).*

Proof. Observe that $\beta(J^+) \cup \beta(J^-) = \beta(J)$. If z is a vertex of $\beta(J^+) \cap \beta(J^-)$, then z is the image of some Σ_{D^g} which lies entirely in a fiber complex ED^g. The only way for $\beta(J^+)$ and $\beta(J^-)$ to intersect in a vertex v is if an edge e' dual to $W_e^{\mathcal{E}X_0}$ contained in Σ_{D^g} collapses. Let EX_v be the fiber complex over v and let $K_v = \mathrm{Stab}_K(v)$. Then the hyperplane $W_{e'}$ in EX_v dual to an edge e' of EX_v is contained in $W_e^{\mathcal{E}X_0}$ and $\mathrm{Stab}_{K_v}(W_{e'}) \le \mathrm{Stab}_K(W_e^{\mathcal{E}G})$ by Proposition 8.12. Since Σ_{D^g} contains e', D^g is commensurable to a subgroup of $\mathrm{Stab}_{K_v}(W_{e'}) \le \mathrm{Stab}_G(W_e^{\mathcal{E}G})$ by Proposition 8.5. Thus $\mathrm{Stab}_K(v)$ is commensurable to a subgroup of $\mathrm{Stab}_K(W_e^{\mathcal{E}G})$ and thus $\Lambda \mathrm{Stab}_K(v) \subseteq \Lambda \mathrm{Stab}_K(W_e^{\mathcal{E}G})$.

Since every path between components of $\mathcal{E}X_0^{(1)} \smallsetminus L$ must traverse an edge dual to $W_e^{\mathcal{E}G}$, every path between components of $\Gamma \smallsetminus \beta(L)$ must intersect $\beta(J^+) \cap \beta(J^-)$. $\qquad\square$

Lemma 8.15. *Let L be the hyperset associated to $W_e^{\mathcal{E}X_0}$. If $D \in \mathcal{D}$ and $x, y \in ED^{(1)} \smallsetminus L$ are distinct points that lie in distinct complementary components of L, then x and y lie in distinct complementary components of $\mathcal{E}X_0^{(1)} \smallsetminus L$.*

Proof. Since $W_e^{\mathcal{E}G}$ splits $\mathcal{E}X_0$ into two components, any path in $\mathcal{E}X_0^{(1)}$ has endpoints in different components of $\mathcal{E}X_0^{(1)} \smallsetminus L$ if and only if the path crosses L an odd number of times. Likewise, complementary components of $ED^{(1)} \smallsetminus W_e^{\mathcal{E}G}$ are determined by the number of times a path crosses the hyperplane $W_e^{\mathcal{E}X_0} \cap ED$. $\qquad\square$

To prove relative quasiconvexity of $\mathrm{Stab}_K(W_e^{\mathcal{E}X_0})$ we need relative quasiconvexity of hyperplane stabilizers of fiber complexes in $(K, \mathcal{R})$.

Proposition 8.16. *Suppose H_W is a hyperplane stabilizer for a hyperplane in a fiber complex ED^g:*

(1) *H_W is relatively quasiconvex in $(D^g, \mathcal{P}_D^g)$*

(2) H_W *is relatively quasiconvex in* $(K, \mathcal{R})$.

Proof. The first claim follows from [EGN24, Corollary 4.11].

Since $(D^g, \mathcal{R}_D^g)$ is the peripheral structure induced by $(K, \mathcal{R})$ and H_W is relatively quasiconvex in $(D, \mathcal{R}_D^g)$, H_W is relatively quasiconvex in $(K, \mathcal{R})$ by [Hru10, Corollary 9.3]. $\qquad\square$

Proposition 8.17. *Let* $D \in \mathcal{D}$. *Let* $p, q \in \partial_{\mathcal{R}} K$ *be distinct so that* $p, q \in \partial_{\mathcal{R}_D} D$. *There exists an edge* e *of* ED *so that if* L *is the hyperset associated to* $W_e^{\mathcal{E} X_0}$, *the hyperset* $\beta(L)$ *separates* p, q *in the sense of* Definition 5.13.

Proof. Choose $W_e^{\mathcal{E} X_0}$ as above so that if $W_D = W_e^{\mathcal{E} X_0} \cap ED$, $\Lambda \operatorname{Stab}_D(W_D)$ separates p, q in $\partial_{\mathcal{R}_D} D$. We describe points in the Bowditch Boundary $\partial_{\mathcal{R}} K$ in terms of the fine hyperbolic graph Γ as described in Definition 5.6. As in the proof of Proposition 6.10, the proof splits into the following three cases: (1) two parabolic points, (2) two conical limit points, and (3) one parabolic and one conical limit point.

Case 1: p, q **are both parabolic vertices of** Γ. If p, q are both parabolic vertices, then p, q already lie in distinct components of $\mathcal{E} X_0 \smallsetminus W_e^{\mathcal{E} X_0}$ because $W_e^{\mathcal{E} X_0}$ intersects ED exactly in the hyperplane dual to e. Also, $p, q \notin \Lambda \operatorname{Stab}_K(W_e^{\mathcal{E} X_0})$ because $p, q \in \partial_{\mathcal{R}_D} D$ and $\Lambda \operatorname{Stab}_D(W_D)$ separates p, q in $\partial_{\mathcal{R}_D} D$. Note that $\beta(p)$ and $\beta(q)$ cannot be in L because otherwise p or q would have to be in $\Lambda \operatorname{Stab}_D(W_D) \subseteq \Lambda \operatorname{Stab}_K(W_e^{\mathcal{E} X_0})$ by [EGN24, Proposition 6.1]. Then $\beta(p), \beta(q)$ are in distinct components of $\Gamma \smallsetminus \beta(L)$.

Setup for the remaining cases: Let γ be a geodesic in Γ between p, q (recall Proposition 5.12 implies existence of γ). Recall that $(K, \mathcal{R})$ admits a generalized fine action on $\mathcal{E} X_0^{(1)}$ Proposition 8.9. By [EGN24, Definition 2.6 and Corollary 2.11], there exists a quasi-geodesic $\widehat{\gamma}$ in $\mathcal{E} X_0^{(1)}$ called the **complete de-electrification** of γ such that $\beta(\widehat{\gamma}) = \gamma$.

Case 2: p, q **are both conical limit points.** Now suppose p, q are both conical limit points. Since W_D separates p, q, there exists a quasi-geodesic $\widehat{\sigma} : (-\infty, \infty) \to ED$ so that $\sigma = \beta(\widehat{\sigma})$ is geodesic in $\beta(ED)$ between p, q and σ has the properties described in Lemma 8.7. By quasi-geodesic stability, there exists an $R_1 \geq 0$ so that $d_{Haus}(\widehat{\sigma}, \widehat{\gamma}) < R_1$.

By Lemma 8.7, for any $M > 0$, there exists $T >> 0$ so that if $|t| > T$, $d_{ED^{(1)}}(W_D, \widehat{\sigma}(t)) > M$.

Claim: for $N > 0$, there exists T_N so that if $|t| > T_N$, $d_{\mathcal{E} X_0^{(1)}}(\widehat{\gamma}(t), W_e^{\mathcal{E} X_0}) > N$.

If there exists a sequence $t_i \to \infty$ so that $d_{\mathcal{E} X_0^{(1)}}(\widehat{\gamma}(t_i), W_e^{\mathcal{E} X_0}) \leq N$, then eventually $\beta(\widehat{\gamma}) = \gamma$ remains within a bounded distance of $\beta(L)$ by quasiconvexity of $\beta(J)$. Then as $t \to \infty$, $\gamma(t)$ converges to either p or q, so one of these lie in $\Lambda \operatorname{Stab}_K(L)$. Suppose without loss of generality that $\gamma(t) \to p$ as $t \to \infty$. Then $p \in \Lambda D \cap \Lambda \operatorname{Stab}_K(L) = \Lambda \operatorname{Stab}_K(W_D)$ by relative quasiconvexity which contradicts the fact that W_D separates p, q.

Claim: for $|t| >> 0$, $\widehat{\gamma}(\pm t)$ lie in distinct components of $\mathcal{E} X_0^{(1)} \smallsetminus L$. For each t, there exist $t'_t \in \mathbb{R}$ so that $d(\widehat{\sigma}(t'_t), \widehat{\gamma}(t)) < R_1$. Since $\widehat{\gamma}, \widehat{\sigma}$ are quasi-geodesics, we can ensure for $|t| >> 0$ that $|t'_t| >> 0$ and $d(\widehat{\gamma}(t), W_D) > R_1$. Thus for $t >> 0$, $\widehat{\gamma}(t)$ and $\sigma(t'_t)$ lie in the same component of $\mathcal{E} X_0 \smallsetminus L$.

Since $\widehat{\sigma}(\pm t')$ eventually lie in distinct components of $\mathcal{E}X_0^{(1)} \smallsetminus L$, for $t >> 0$, $\widehat{\gamma}(\pm t)$ also lie in distinct components of $\mathcal{E}X_0^{(1)} \smallsetminus L$.

For any $D \in \mathcal{D}$ and $g \in K$, the sub-graph whose cells are stabilized by Σ_{D^g} has finite diameter. So if Σ_{D^g} has an edge dual to $W_e^{\mathcal{E}X_0}$, then for $t >> 0$, $\widehat{\gamma}(\pm t)$ cannot lie in Σ_{D^g} by Lemma 8.7. Therefore, for all $t_- << 0$ and $t_+ >> 0$ $\beta(\widehat{\gamma}(t_-)), \beta(\widehat{\gamma}(t_+))$ lie in distinct components of $\Gamma \smallsetminus \beta(L)$. Since $\beta(\widehat{\gamma}) = \gamma$, $\beta(L)$ separates p, q.

Case 3: p is a conical limit point and q is a parabolic vertex. Since W_D separates p, q, there exists a quasi-geodesic $\widehat{\sigma} : (-\infty, \infty) \to ED$ so that $\sigma = \beta(\widehat{\sigma})$ is geodesic in $\beta(ED)$ between p, q and σ has the properties described in Lemma 8.8. In this case, parameterize $\sigma : [0, \infty) \to \Gamma$, $\widehat{\sigma} : [0, \infty) \to \mathcal{E}X_0^{(1)}$, $\widehat{\gamma} : [0, \infty) \to \mathcal{E}X_0^{(1)}$ so that $\gamma(0) = \sigma(0) = \beta(\widehat{\gamma}(0)) = q$. By Lemma 8.8, for $t >> 0$, $\widehat{\sigma}(t)$ and $\widehat{\sigma}(0)$ lie on different sides of W_D and $d(\widehat{\sigma}(t), W_D)$ can be made arbitrarily large. Following an argument similar to the one in the previous case, for $t' >> 0$, $\widehat{\gamma}(t')$ can be made arbitrarily far from $W_e^{\mathcal{E}X_0}$ so that $p \notin \Lambda \operatorname{Stab}_K(L)$, and $\widehat{\gamma}(t)$ lies in the same component of $\mathcal{E}X_0^{(1)} \smallsetminus L$ as $\widehat{\sigma}(t')$ for $t, t' >> 0$. Note that $\beta(q)$ cannot lie in $\beta(L)$ by an argument similar to the one in Case 1. Hence for $t'' >> 0$, $\gamma(t'')$ and $\gamma(0) = q$ must lie in distinct components of $\Gamma \smallsetminus \beta(L)$. $\qquad\square$

While D was used in Proposition 8.17 to simplify notation, D can be replaced by the (maximal parabolic in $(K, \mathcal{D})$) stabilizer of any fiber complex without issue. We now show that pairs of points in $\partial_{\mathcal{R}} K$ with the same image in $\partial_{\mathcal{D}} K$ can be separated by some $\mathcal{E}X_0$–hyperstructure associated to an edge in the corresponding fiber complex.

Theorem 8.18. *Suppose $(\mathcal{E}X_0, \pi)$ satisfies the projected wall tree, projected wall fullness and two-sided wall projection properties. Let ED^g be a fiber complex over v and let $(D^g, \mathcal{R}_D^g)$ be the induced peripheral structure on the stabilizer D^g of ED^g. Let $p, q \in \partial_{\mathcal{R}_D^g} D^g \subseteq \partial_{\mathcal{R}} K$ be distinct, then there exist $W_e^{\mathcal{E}X_0}$ for some edge e of ED^g so that for some finite index $H_W \leq \operatorname{Stab}_K(W_e^{\mathcal{E}X_0})$, p and q are in H_W–distinct components of $\partial_{\mathcal{D}} K \smallsetminus H_W$.*

Proof. By Proposition 8.17 there exists a $W_e^{\mathcal{E}X_0}$ that separates p, q. The action of K on Γ has trivial edge stabilizers, the action is cocompact and every maximal parabolic fixes a vertex. Therefore, by Theorem 5.15 (using Proposition 8.14 to ensure the hypotheses hold), there exists a finite index subgroup H_W of $\operatorname{Stab}_K(W_e^{\mathcal{E}X_0})$ so that p and q lie in H_W-distinct components of $\partial_{\mathcal{R}} K \smallsetminus \Lambda H_W$. $\qquad\square$

8.3. **Applying the boundary criterion.** We now prove a more general theorem that will imply Theorem 1.2 after proving $\mathcal{E}G_{bal}$ (together with the natural projection) is a relatively geometric blowup of $(X_{bal}, K, \mathcal{D})$.

We will use the boundary criterion for relatively geometric cubulation criterion.

Theorem 2.7. ([EMN24, Theorem 1.3], *see also* [EG20, Theorem 2.6]) *Let $(G, \mathcal{P})$ be a finitely generated relatively hyperbolic and suppose that for every pair of distinct points p and q in the Bowditch Boundary $\partial_{\mathcal{P}} G$ there is a full relatively quasiconvex codimension 1 subgroup H of G so*

that p and q lie in H–distinct components of $\partial G \smallsetminus \Lambda H$. Then there exist finitely many full relatively quasiconvex codimension–1 subgroups of G and a refined peripheral structure $(G, \mathcal{P}')$ for $(G, \mathcal{P})$ so that the action of $(G, \mathcal{P}')$ on the dual cube complex is relatively geometric.

Moreover, if each $P \in \mathcal{P}$ acts elliptically on the dual cube complex then no refinement is needed, i.e., we may take $\mathcal{P}' = \mathcal{P}$.

The following is a naive application of Theorem 2.7 and subsequent refinements result in stronger consequences on the peripheral structures.

Theorem 8.19. *If $(\mathcal{E}X_0, \pi)$ is a relatively geometric blowup of $(X_0, K, \mathcal{D})$ that satisfies the projected wall tree, projected wall fullness and two-sided wall projection properties (recall Definition 8.10, then there is a refined peripheral structure $(K, \mathcal{R}')$ of $(K, \mathcal{D})$ and $(K, \mathcal{R})$ so that $(K, \mathcal{R}')$ acts relatively geometrically on a CAT(0) cube complex.*

Proof of Theorem 8.19. Let $p, q \in \partial_{\mathcal{R}} K$ be distinct points in the Bowditch Boundary. Let $\mathcal{R}_D^g$ be the induced peripheral structure on a fiber complex stabilizer D^g. By [Yan14, Theorem 1.1], the fiber complex stabilizers are relatively quasiconvex in $(K, \mathcal{R})$. Thus $\partial_{\mathcal{R}_D} D^g$ embeds in $\partial_{\mathcal{R}} K$. Recall that $(K, \mathcal{D})$ is a relatively hyperbolic pair, so fiber complex stabilizers have pairwise finite intersections. Thus Bowditch boundaries with respect the induced peripheral structures of distinct fiber groups embed as disjoint subspaces in $\partial_{\mathcal{R}} K$.

If p, q both lie in $\partial_{\mathcal{R}_D^g} D^g$, then Theorem 8.18 implies that there exists a full relatively quasiconvex (in $(K, \mathcal{R})$ $H \le K$ so that p, q lie in H–distinct components of $\partial_{\mathcal{R}} K \smallsetminus \Lambda H$.

On the other hand if p, q are not both in the boundary of a single fiber group, then Corollary 8.6 implies that there exists a full relatively quasiconvex $H_{p,q} \le K$ so that p, q lie in $H_{p,q}$–distinct components of $\partial_{\mathcal{R}} K \smallsetminus \Lambda H_{p,q}$.

Then, Theorem 2.7 implies that there exists a refined peripheral structure $(K, \mathcal{R}')$ for $(K, \mathcal{R})$ so that $(K, \mathcal{R}')$ acts relatively geometrically on a CAT(0) cube complex. $\square$

Let us now verify that the moreover statement of Theorem 2.7 holds, so no refinement is needed.

Proposition 8.20. *Suppose that $(\mathcal{E}X_0, \pi)$ is a relatively geometric blowup of $(X_0, K, \mathcal{D})$ that satisfies Definition 8.10 and $(K, \mathcal{R})$ is the associated refined peripheral structure. Each peripheral subgroup $R \in \mathcal{R}$ acts elliptically on the dual cube complex.*

Lemma 8.21. *Suppose that W is any $\mathcal{E}X_0$–hyperstructure and $H_W < \mathrm{Stab}_K(W)$ is the index 2 subgroup without inversion in W. A choice of complementary component determines a halfspace of some sufficiently large neighborhood of H_W in K.*

Proof. By construction hyperstructures are two-sided. There is an equivariant quasi-isometry between $\mathcal{E}X_0$ and the coned-off Cayley graph $\Gamma(G, \mathcal{R})$ by Charney and Crisp [CC07, Theorem 5.1], so W–complementary components induce (coarse) halfspaces of H_W each containing a neighborhood of $\mathrm{Stab}_K(W)$. Moreover, these halfspaces are H_W–invariant since H_W does not invert W.

To obtain coarse halfspaces in K, one verifies that unconing $\Gamma(G, \mathcal{R})$ leaves the halfspaces unchanged. Indeed, if Σ_{R^g} is the subgraph will cell stabilizers commensurable to R^g (as in Proposition 8.9) is disjoint from W then R cannot identify large neighborhoods of H_W because it has deep intersection with at most one halfspace of H_W. On the other hand, if Σ_{R^g} intersects W then by fullness $H_W \cap R^g$ is finite index in R^g, so a sufficiently large neighborhood of H_W will be contained in a single halfspace. $\qquad\square$

Proof of Proposition 8.20. If any $R \in \mathcal{R}$ is finite then it already acts elliptically since isometries of finite dimensional CAT(0) cube complexes are semisimple [Hag08]. Up to cubical subdividing we can and will assume that elliptic subgroups fix a vertex.

By definition of relatively geometric blowup, up to conjugation any peripheral subgroup stabilizes a vertex x contained in some fiber complex $\pi^{-1}(v)$. Let K_x denote the stabilizer of x.

By construction, each hyperstructure (in either X_0 or $\mathcal{E}X_0$) is disjoint from vertices, so its complementary components determine a bipartition of the vertices.

For each X_0–hyperstructure, choose the complementary component containing v. For each $\mathcal{E}X_0$–hyperstructure choose the complementary component that contains x. The above choices determine a principal ultrafilter and hence a vertex of the dual cube complex. Indeed, the cube complex obtained using Theorem 2.7 is dual to stabilizers of hyperstructures in X_0 and $\mathcal{E}X_0$ and Lemma 8.21 demonstrates that choices of halfspaces are determined by choices of complementary components of the corresponding hyperstructure.

Finally, K_x stabilizes x in $\mathcal{E}X_0$ and hence v in X_0. Thus, no element of K_x will flip any of the halfspaces. Therefore, K_x fixes a vertex in the dual cube complex. $\qquad\square$

Corollary 8.22. *Suppose that $(\mathcal{E}X_0, \pi)$ is a relatively geometric blowup of $(X_0, K, \mathcal{D})$ that satisfies Definition 8.10 and $(K, \mathcal{R})$ is the associated refined peripheral structure. Then K admits a relatively geometric action on a CAT(0) cube complex with respect to either $(K, \mathcal{D})$ or $(K, \mathcal{R})$.*

Proof. The relative cubulation for $(K, \mathcal{R})$ is immediate from Theorem 8.19 and Proposition 8.20. For $(K, \mathcal{D})$, recall from Remark 8.2 that X_0 is a relatively geometric blowup of itself where the refined peripheral structure is $(K, \mathcal{D})$. $\qquad\square$

9. Relative Cubulations for Small Cancellation Free Products of Relatively Cubulable Groups.

9.1. $\mathcal{E}G_{bal}$ is a blowup.

We now verify that if $p : \mathcal{E}G_{bal} \to X_{bal}$ is the natural projection from Section 4.2, then $(\mathcal{E}G_{bal}, p)$ is a relatively geometric blowup of $(X_{bal}, K, \mathcal{D})$.

We previously verified Hypotheses 6.2 in Section 7. We now verify Definition 8.1 and Definition 8.10 item by item:

The cell structure and even-sided condition follow from the choice of subdivision in the definition of $\mathcal{E}G_{bal}$, and simply connectedness and cocompactness of the G action follow from Proposition 3.12 verifying (1) and (2).

Items (3) and (4) follow from the description of the fibers of $\mathcal{E}G$ given in Section 3.4 and the fact that the subdivision of $X, \mathcal{E}G$ to form $X_{bal}, \mathcal{E}G_{bal}$ takes polygons to polygons and cubes to cubes.

Item (5) is also immediate from the construction in Section 3.4 in that the fibers over the vertices stabilized by A, B are specifically constructed to be copies of EA and EB admitting relatively geometric actions of $(A, \mathcal{P}_A)$ and $(B, \mathcal{P}_B)$ respectively.

We show that $(\mathcal{E}G_{bal}, p)$ satisfies each Items (6),(7),(8) of Definition 8.10.

Proposition 9.1. *The blowup* $(\mathcal{E}G_{bal}, p)$ *has the wall tree projection property, the projected wall fullness property and two-sided wall projection property.*

Proof. The wall tree projection property follows from [MS17, Lemma 3.32]. The projected wall fullness property follows from Proposition 7.2.

To see that the two-sided wall projection property hold consider U and U^* the two closed halfspaces of an arbitrary wall $W = W_e^{\mathcal{E}G}$. It suffices to show that $p(U) \cap p(U^*) = p(U \cap U^*)$.

Let $x \in p(U) \cap p(U^*)$. By Definition 8.1, the projection $p : \mathcal{E}G_{bal} \to X_{bal}$ restricts to a homeomorphism on the preimage of each open 1–cell or 2–cell of X_{bal}, moreover, the preimage of each 0–cell is a (connected) fiber complex.

Hence, if x is contained in the interior of a 1–cell or 2–cell then there is a unique point $\tilde{x} \in p^{-1}(x)$ and it must be that $\tilde{x} \in U \cap U^*$. Otherwise, x is a 0–cell of X_{bal} with preimage E_x a fiber complex. Since $x \in p(U)$ it must be that $E_x \cap U \neq \varnothing$, and similarly, $E_x \cap U^* \neq \varnothing$. Since U, U^* are half spaces of W that both intersect E_x, $U \cap E_x$, $U^* \cap E_x$ are half spaces of a hyperplane $W \cap E_x$ in E_x. Thus there exists a point $\tilde{x} \in E_x \cap (U \cap U^*)$ such that $p(\tilde{x}) = x \in p(U \cap U^*)$ as desired. $\qquad\square$

9.2. **The proof of Theorem 1.2.**   We are now ready to prove Theorem 1.2:

Theorem 1.2. *Let* $(A_i, \mathcal{P}_i)$ *be a collection of* $n \geq 2$ *finitely generated groups that each act relatively geometrically on* $\mathrm{CAT}(0)$ *cube complexes, and let* $\mathcal{P} = P_1 \cup \cdots \cup P_n$. *If* G *is a* $C'(\frac{1}{6})$–*free product of* $A_1, \ldots, A_n$ *by a finite set of relators, then* $(G, \mathcal{P})$ *acts relatively geometrically on a* $\mathrm{CAT}(0)$ *cube complex.*

Proof. Construct the complexes X_{bal} and $\mathcal{E}G_{bal}$ as constructed in Section 4 with a natural combinatorial projection $pr : \mathcal{E}G_{bal} \to X_{bal}$.

In Section 9.1, we show that $(\mathcal{E}G_{bal}, \pi)$ is a relatively geometric blowup of $(X_{bal}, G, \mathcal{Q})$, which gives G the peripheral structure $(G, \mathcal{P})$. In Proposition 9.1, we show the remaining properties in Definition 8.10 hold. Then Corollary 8.22 implies the result. $\qquad\square$

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